# Exact Critical Behavior of a Random-Bond Two-Dimensional Ising Model

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A 2D Ising model in which the bonds $K$ fluctuate randomly about $K_c$, the critical value of the pure system, is considered. The ensemble average of the *square* of the two-point function, $\langle\langle s_0 s_R\rangle^2\rangle_{\rm Av}$, is shown to decay as $(\ln R)^{1/4}R^{-1/2}$ at the critical point. This implies that $\langle\langle s_0 s_R\rangle\rangle_{\rm Av}$ is bounded above by $(\ln R)^{1/8}R^{-1/4}$ in disagreement with the $\exp[-(\ln\ln R)^2]$ decay law found by Dotsenko and Dotsenko by a different method. On the other hand, the present calculation reproduces their specific-heat singularity $C\sim\ln|\ln\tau|$ $(\tau=K-K_c)$.

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Since Onsager's solution of the Ising model, we have come a long way in terms of exact solutions for two-dimensional (2D) homogeneous statistical mechanics models, at the level of not only free energies but also correlation functions.[1,2] In contrast there are few exact results in 2D for the averaged free energies and fewer results for the averaged correlation functions of random systems, where the interactions or bonds are drawn from a probability distribution. McCoy and Wu,[3] who evaluated the free-energy singularity for an Ising model with correlated randomness in the bonds, also evaluated correlations of boundary spins. Recently Shankar and Murthy[4] calculated bulk correlation functions for a related model that allowed for frustration. There is an inequality for the correlation-length index $\nu$ due to Chayes *et al.*[5]

We are concerned here with a nearest-neighbor problem where the bonds $K$ fluctuate *independently* at each site about the critical value $K_c$ of the pure system. This problem was investigated by Dotsenko and Dotsenko (DD).[6] Using continuum methods they found that the specific heat $C(\tau)$ has a $\ln|\ln\tau|$ singularity ($\tau$ being $K-K_c$) in contrast to the $\ln\tau$ singularity of the pure case. For the ensemble-averaged spin correlation $\langle\langle s_0 s_R\rangle\rangle_{\rm Av}$, they found a decay law $\exp[-(\ln\ln R)^2]$ in contrast to the $R^{-1/4}$ decay in the pure case. Since they show that randomness is marginally attractive one expects the pure-system behavior to be modified at most by logarithms, as in the case of the free energy and in contrast to the correlation function whose index $\eta$ has changed from $\frac{1}{4}$ to 0. I was therefore motivated to reexamine the results. Using a different and simpler procedure, I found that the ensemble-averaged *square* of the correlation function decays as $(\ln R)^{1/4}R^{-1/2}$. Now the average over samples of the square of the correlation must be larger than the square of a similar average. It follows that $\eta\geq\frac{1}{4}$. Since this result contradicts DD, I will provide the details of my calculation and follow this with a derivation of the $\ln|\ln\tau|$ singularity in $C$ using my approach.

Since both our calculations rely heavily on the fermionic representation of the Ising model, a few basic notions will now be recalled. Let us begin with the transfer matrix for the pure system with bonds $K$ (ignoring analytic prefactors):

$$T=\exp[K^*\textstyle\sum_n\sigma_1(n)]\exp[K\sum_n\sigma_3(n)\sigma_3(n+1)],\tag{1}$$

where $K^*=-[\ln\tanh K]/2$ and $\sigma(n)$ are Pauli matrices at site $n$ of the row. Let us now introduce Hermitian (i.e., Majorana) fermion operators

$$\psi_1(n)=2^{-1/2}\prod_{-\infty}^{n-1}\sigma_1(m)\sigma_2(n),\tag{2}$$

$$\psi_2(n)=2^{-1/2}\prod_{-\infty}^{n-1}\sigma_1(m)\sigma_3(n),\tag{3}$$

which obey the anticommutation rules

$$\{\psi_i(n),\psi_j(m)\}_+=\delta_{ij}\delta_{mn}.\tag{4}$$

Dirac fermions used by Schultz, Mattis, and Lieb[7] are given by

$$\Psi(n)=[\psi_1(n)+i\psi_2(n)]/\sqrt{2}\tag{5}$$

and obey the more familiar anticommutation rules. Since

$$\begin{aligned}\sigma_1(n)&=-2i\psi_1(n)\psi_2(n),\\ \sigma_3(n)\sigma_3(n+1)&=2i\psi_1(n)\psi_2(n+1),\end{aligned}\tag{6}$$

we are dealing with a free-fermion theory. When $\tau=K-K^*$ tends to zero, we obtain in the continuum limit a Majorana field theory with mass $m=\tau$. We may represent the partition function (up to some prefactors) as a Grassmann integral in Euclidean space[8]:

$$Z=\int d\psi\exp\left\{\frac{1}{2}\int[\bar\psi\boldsymbol{\partial}\psi+\tau\bar\psi\psi]d^2x\right\}.\tag{7}$$

Expressing the fermion integral as the square root of $\det[\boldsymbol{\partial}+\tau]$ and using the identity $\det A=\exp\operatorname{tr}\ln A$ one obtains the usual $\tau^2 \ln\tau$ singularity in the free energy. The correlation function poses a far greater challenge, as can be seen from Eq. (6). Since

$$\sigma_3(0)\sigma_3(R)=\sigma_3(0)\sigma_3(1)\sigma_3(1)\sigma_3(2)\cdots\sigma_3(R)\propto\psi_1(0)\left\{\exp\left[i\pi\sum_{m=1}^{R-1}\psi_2(m)\psi_1(m)\right]\right\}\psi_2(r), \tag{8}$$

we find that $\langle s_0 s_R\rangle$ is given by the vacuum expectation of the above string of fermions (*whose length grows with the separation*) so that its evaluation is very tedious in the operator formalism.[7] DD express it in the functional integral formalism as the average of a similar line integral with respect to the Grassmann measure in Eq. (7) and succeed, using their considerable skills, in evaluating it at $\tau=0$ to obtain $\eta=\frac{1}{4}$. They then generalize all this to the random case using the replica trick.[9] $Z$ is now given by an O($n$) Gross-Neveu model[10] at $n=0$ with a mass term proportional to $\tau$ and four-fermion coupling $\frac{1}{2}g^2(\bar\psi\psi)^2$, where $g^2$ measures the variance of the bond distribution. [We can understand this as follows. If we replace $\tau$ by $\tau(x)$ in Eq. (7) to represent the fluctuating bonds, use the replica trick,[9] and average $\tau(x)$ over a Gaussian distribution of mean $\tau$ and variance $g^2$ the result follows.] At $\tau=0$ the fermion string is evaluated by DD by renormalization-group (RG) methods.

We will follow a different route inspired by the work of Itzykson and Zuber (IZ).[11] For reasons that will be apparent soon, these authors consider the *square* of the two-spin correlation function for the *pure* system. Squaring introduces a second Majorana fermion $\chi$ which combines with $\psi$ to form a Dirac fermion $\Psi=(\psi+i\chi)/\sqrt{2}$, while the fermion string essentially becomes the line integral of the fermion charge $j_0$:

$$\langle s_0 s_R\rangle^2 = Z^{-1}\int d\bar\Psi d\Psi \exp\left\{\int[\bar\Psi\boldsymbol{\partial}\Psi+\tau\bar\Psi\Psi]d^2x\right\}\exp\left(i\pi\int_0^{x=Ra} j_0\,dx'\right), \tag{9}$$

$$Z=\int d\bar\Psi d\Psi\exp\left\{\int[\bar\Psi\boldsymbol{\partial}\Psi+\tau\bar\Psi\Psi]d^2x\right\},$$

where $a$ is the inverse of the momentum cutoff $\Lambda$. (Thus, a separation of $R$ lattice sites corresponds to a laboratory distance $x=Ra$.)

To understand Eq. (9) note that Eq. (8) now is replaced by

$$\sigma_3(0)\mu_3(0)\sigma_3(R)\mu_3(R)\propto\psi_1(0)\chi_1(0)\left[\prod_i^{R-1}\psi_2\chi_2\psi_1\chi_1\right]\psi_2(R)\chi_2(R), \tag{10}$$

where the $\mu$'s are a set of Pauli matrices identical to the $\sigma$'s. If we now exponentiate the product in the middle as before and recall that $j_0=\Psi^\dagger\Psi$, Eq. (9) follows, except for the end factors at 0 and $R$ which will be put back eventually, and powers of $a$ that relate lattice quantities to their continuum counterparts.

The point of doing all this is that given a Dirac fermion, we can bosonize,[12] which (in the Euclidean version) amounts to the replacement

$$\bar\Psi\boldsymbol{\partial}\Psi=-\tfrac{1}{2}(\nabla\phi)^2, \tag{11}$$

$$\bar\Psi\Psi=\Lambda\cos(4\pi)^{1/2}\phi, \tag{12}$$

$$j_\mu=\bar\Psi\gamma^\mu\Psi=\pi^{-1/2}\epsilon_{\mu\nu}\partial\phi/\partial x_\nu. \tag{13}$$

Those who are unfamiliar with bosonization need only be aware that the Green's functions of any string of fermionic operators in the left-hand side of Eq. (12) or (13) with respect to the fermionic action density in Eq. (11) will coincide with that of the corresponding bosonic string with the bosonic action in Eq. (11). To the extent that interacting theories can be defined by their perturbation series about the free-field limit, bosonization allows us to pass from the fermionic to the bosonic version and vice versa using these rules. The last equation implies

$$\int_0^{Ra} j_0(x)\,dx=\pi^{-1/2}\int_0^{Ra}d\phi =[\phi(Ra)-\phi(0)]/\sqrt{\pi}, \tag{14}$$

so that the correlation function squared becomes a *two-point function* of the operator $\exp(i\pi^{1/2}\phi)$ in the sine-Gordon theory. If one keeps the end factors one finds that the operator in question is[11]

$$O=(a\mu)^{1/4}N_\mu(\sin\pi^{1/2}\phi), \tag{15}$$

where $N_\mu$ denotes normal ordering at mass $\mu$. Since[12]

$$N_\mu[e^{i\beta\phi}]=\{\Lambda/\mu\}^{\beta^2/4\pi}e^{i\beta\phi}, \tag{16}$$

we find that $O$ is just the unordered operator $\sin\pi^{1/2}\phi$. At the critical point the cosine interaction vanishes and all one needs are Gaussian integrals. (See, for example, Ref. 12 for their evaluation.) Evaluating them, IZ obtain the result

$$\langle s_0 s_R\rangle^2=[1/\Lambda Ra]^{1/2}=[1/R]^{1/2} \tag{17}$$

since $a\Lambda=1$. When we take the square root, $\eta=\frac{1}{4}$ follows.

We will now generalize this bosonic picture to the random case. For a given realization of bonds,

$$\langle s_0 s_R\rangle^2 \sim \int O(0)O(Ra)e^{S(\phi)}\,d\phi/Z_B \quad [Z_B=\int e^{S(\phi)}d\phi], \tag{18}$$

$$S(\phi)=\int[-\tfrac{1}{2}(\nabla\phi)^2+\tau(x)\Lambda\cos(4\pi)^{1/2}\phi]\,d^2x. \tag{19}$$

We multiply the numerator and the denominator of (18) by $Z_B^{n-1}$, and set $n=0$. Next we average over $\tau(x)$ using a Gaussian of mean 0 (to be at the critical point) and variance $2g^2$. The result is

$$\langle\langle s_0 s_R\rangle^2\rangle_{\rm Av}=\int O_1(0)O_1(x=Ra)e^{S(n,\phi)}\,d\phi, \tag{20}$$

where the subscript 1 tells us that it is the replica we began with and

$$S(n,\phi)=\int\left\{\sum_{i=1}^{n}[-\tfrac{1}{2}(\nabla\phi_i)^2]+\frac{g^2}{2}\Lambda^2\sum_{i=1}^{n}\sum_{j=1}^{n}\cos(4\pi)^{1/2}\phi_i\cos(4\pi)^{1/2}\phi_j\right\}d^2x. \tag{21}$$

Next we eliminate the $i=j$ terms in the double sum using the identity

$$[\Lambda\cos(4\pi)^{1/2}\phi]^2=-(2\pi)^{-1}(\nabla\phi)^2 \tag{22}$$

which can be derived[13] either naively, recalling that in $d=2$, $[\bar\Phi\gamma^\mu\Phi]^2=-2[\bar\Phi\Phi]^2$, and then bosonizing, or by use of the operator-product expansion. The effect of this is to rescale the coefficient of the kinetic term by a factor $1+g^2/2\pi$. If we rescale $\phi$ by the square root of this factor to obtain the standard kinetic term, the operator $O_1$ changes:

$$O_1(x)=\sin\{[\pi/(1+g^2/2\pi)]^{1/2}\phi_1(x)\}, \tag{23}$$

and the cosines in Eq. (21) are likewise modified. (The power of bosonization can be appreciated if one notes that in the Ashkin-Teller model, where $g^2$ is the four-spin coupling, these manipulations would give us the exponents varying continuously with $g^2$.)

We now prepare to apply renormalization-group ideas to the $O_1$-$O_1$ correlation function, $\Gamma(x,g,\Lambda)$, which obeys[13]

$$\Gamma(x,g,\Lambda)=\exp\left[2\int_g^{g(x\Lambda)}\frac{\gamma(g)}{\beta(g)}\,dg\right]\Gamma(1,g(x\Lambda)). \tag{24}$$

In the above, the $\beta$ function of the Gross-Neveu model is known:

$$\beta(g)=[\partial g/\partial\ln\Lambda]_{g(\mu)}=-[n-1]g^3/2\pi, \tag{25}$$

where $g(\mu)$ is the renormalized coupling at scale $\mu$. As for $\gamma(g)$, it is

$$\gamma(g)=[\partial\ln Z_0/\partial\ln\Lambda]_{g(\mu)}, \tag{26}$$

where $Z_0(g,\Lambda/\mu)$ multiplicatively removes the $\Lambda$ dependence of $O_1$ correlation functions. Let us now derive $\gamma$ to order $g^2$. At the lowest order the $O_1$-$O_1$ function goes as $[x\Lambda]^{-1/2}$. Since rescaling $O$ by $\Lambda^{1/4}$ renormalizes it, $\gamma=\frac{1}{4}$. To order $g^2$, there are two possible sources. First, the definition of $O_1$ itself involves $g^2$; see Eq. (23). In addition there can be a $g^2$ term from our expanding the interaction. This term, which no longer has an $i=j$ piece, is forbidden from contributing to the $O_1$-$O_1$ correlation by the symmetry of the free-field action under independent translation of each $\phi_i$. Thus we are left with the free-field evaluation of $\langle O_1(0,g)O_1(x,g)\rangle$:

$$\langle O_1(0)O_1(x)\rangle\propto\left[\frac{1}{x\Lambda}\right]^{[2(1+g^2/2\pi)]^{-1}}\approx\left[\frac{1}{x\Lambda}\right]^{1/2}\left[1+\frac{g^2}{4\pi}\ln x\Lambda\right]. \tag{27}$$

Clearly,

$$Z_0=\Lambda^{1/4}\left[1-\frac{g^2}{8\pi}\ln\Lambda\right],\quad \gamma(g)=\frac{1}{4}-\frac{g^2}{8\pi}, \tag{28}$$

and Eq. (24) implies

$$\Gamma(x=Ra,g)\propto R^{-1/2}e^{-(1/2)\ln[g(x)/g]}\Gamma(1,g(x)). \tag{29}$$

Integrating Eq. (25) (for the case $n=0$) gives

$$g^2(x\Lambda)=\frac{g^2}{1+[g^2/\pi]\ln x\Lambda}. \tag{30}$$

Substituting this into Eq. (29) we find the advertised result:

$$\langle\langle s_0,s_R\rangle^2\rangle_{\rm Av}\approx(1+[g^2/\pi]\ln R)^{1/4}/R^{1/4}. \tag{31}$$

Since $\langle\langle s_0 s_R\rangle^2\rangle_{\rm Av}\geq[\langle\langle s_0 s_R\rangle\rangle_{\rm Av}]^2$ it follows that $\eta\geq\frac{1}{4}$.

Now a quick derivation of the specific-heat singularity using the present approach. First we square both sides of Eq. (7), which merely doubles the free energy and replaces $\psi$ by $\Psi$. Since $\bar\Psi\Psi$ couples to $\tau$, the average of $\tau(x)$, the specific heat of a given sample is given by the $\bar\Psi\Psi$-$\bar\Psi\Psi$ correlation at zero external momentum. If we now average over $\tau(x)$, and use the replica trick, we are left with the $\bar\Psi\Psi_1$-$\bar\Psi\Psi_1$ correlation function (where 1 is the replica index) in a Gross-Neveu model with $n$ Dirac fermions. Upon bosonization, and use of Eq. (22), $\bar{\Psi}\Psi_1$ will be given by

$$\hat{O}_1 = \Lambda \cos\{[4\pi/(1+g^2/2\pi)]^{1/2}\phi_1\}. \quad (32)$$

Since the cosine is renormalized by normal ordering, Eq. (16) tells us (upon taking into account the prefactor $\Lambda$ that comes with $\hat{O}$) that

$$Z_{\bar{\Psi}\Psi} = \Lambda^{(1+g^2/2\pi)^{-1}-1}, \quad (33)$$

which implies $\hat{\gamma}(g) = -g^2/2\pi$. However $C$, unlike $\Gamma$ in Eq. (24), obeys an inhomogeneous RG equation just like its counterpart in $d=4$. (This is because $Z_{\bar{\Psi}\Psi}$ will remove $\Lambda$ dependence in all except the autocorrelation function which needs a subtraction even in free-field theory.) Following Brézin in Ref. 13 (p. 372) the leading singularity is

$$C(\tau,g) \sim \int_1^{\lambda} \frac{dz}{z} B(g(z)) \exp\left[2\int_1^{z} \frac{dy}{y}\hat{\gamma}(g(y))\right], \quad (34)$$

where $B$, to lowest order, is just a constant (proportional to the coefficient of $\ln\Lambda$ in the free-field limit of the $\bar{\Psi}\Psi_1$-$\bar{\Psi}\Psi_1$ correlation function), $\hat{\gamma}$ is the anomalous dimension function for $[\bar{\Psi}\Psi]_1$, and $\lambda$ is chosen such that the running mass $\tau(\lambda)$ equals a constant, say unity. As for $\tau(\lambda)$ we need only its lowest-order behavior: $\tau(\lambda) = \tau\lambda$. Eq. (34) then tells us that

$$C(\tau,g) \sim \int_1^{1/\tau} \frac{dz}{z} B(0) \exp\left[-\int_1^{z} \frac{dy}{y}\frac{g^2(y)}{\pi}\right]. \quad (35)$$

If we now recall Eq. (30) we get

$$C(\tau,g) \sim \left[\frac{\pi}{g^2}\right] \ln\left[1+\frac{g^2}{\pi}\ln\left(\frac{1}{\tau}\right)\right], \quad (36)$$

in agreement with DD. Notice again the crossover from pure to random system behavior, which occurs at $\tau = \exp[-\pi/g^2]$.

To conclude, I have calculated exactly the asymptotic decay law, $(\ln R)^{1/4}R^{-1/2}$, for the averaged *square* of the correlation function at the critical point, as well as the $\ln|\ln\tau|$ singularity in $C(\tau)$, the average specific heat, for an Ising model where the bonds fluctuate about the critical value of the pure system. Bosonization was freely used and greatly simplified the correlation calculation by converting the fermionic string to a two-point function in the bosonic version, making the application of RG methods straightforward. The resulting bound $\eta \gtrsim \frac{1}{4}$ for the average correlation function contradicts the DD result while the specific-heat calculation is in agreement with theirs. Assuming that the results quoted here are correct we can follow two routes. The first is to pursue the Dotsenko approach till it yields a decay law that agrees with the bound. The other is to argue as follows that the correlation function, as $R \to \infty$, is sample independent up to corrections of order $1/\ln R$. If one evaluates the average of the fourth power of the correlation along the same lines as above, one finds that $\gamma(g)$ *is the same as before to order* $g^2$—the interaction term still does not contribute for the same reason as before—and the average of the fourth power comes out to be the square of the average of the second power. [There are differences, but they are proportional to $g^2(R) \sim 1/\ln R$.] In fact, this behavior holds for all even powers, suggesting that the squared correlation is sample independent asymptotically. If this is so, so must be the correlation and we get its average by simply taking the square root of my answer and obtaining the decay $(\ln R)^{1/8}R^{-1/4}$.

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