Localization in Optics: Quasiperiodic Media

Mahito Kohmoto, Bill Sutherland, and K. Iguchi
Department of Physics, University of Utah, Salt Lake City, Utah 84112
(Received 4 March 1987)

An experiment to probe the (quasi)localization of the photon is proposed, in which optical layers are constructed following the Fibonacci sequence. The transmission coefficient has a rich structure as a function of the wavelength of light and, in fact, is multifractal. For particular wavelengths for which the resonance conditions is satisfied, the light propagation has scaling with respect to the number of layers, as well as an interesting fluctuation.

PACS numbers: 42.20.−y, 71.55.Jv

Localization of electronic states due to disorder is one of the most active fields in condensed-matter physics.[1] Recently, it has been recognized that quasiperiodic systems also could lead to localization.[2] In a quasiperiodic system two (or more) incommensurate periods are superposed, so that it is neither a periodic nor a random system and could be considered to be intermediate between the two.

A particularly interesting quasiperiodic Schrödinger equation in one dimension was proposed by Kohmoto, Kadanoff, and Tang[3] and by Ostlund *et al.*[4] This model is based on the Fibonacci sequence which is constructed recursively as $S_{j+1}=\{S_{j-1},S_j\}$, for $j\geq 1$, with $S_0=\{B\}$ and $S_1=\{A\}$, and so one has $S_2=\{BA\}$, $S_3=\{ABA\}$, $S_4=\{BAABA\}$, and so forth.

The most striking feature of this model is that all the states are critical. Namely, the wave functions are not localized exponentially but only weakly localized and have a rich structure including scaling.[4,5] Also the electrical resistance is bounded by a power law with respect to sample size[6] in contrast to the exponential growth for the localized states. The energy spectrum also has a rich structure; it is a Cantor set with zero Lebesgue measure. Namely, if one picks an energy, it is in a gap with probability 1 and the gaps are dense. Also there are no isolated points. The spectrum has a self-similar structure with various scaling indices (multifractal).[5]

There are some experiments for observing the exotic behavior mentioned above using semiconductor superlattices.[7] However, these systems possess various additional effects, and it is therefore rather difficult to purely observe the effects of quasiperiodicity.

In this paper, we propose an optical experiment with quasiperiodic layers. In this system the one-dimensional theory is strictly valid. Also, it is feasible to construct the system accurately and the parameter may be precisely controlled and measured. Although Anderson localization occurs in a quantum-mechanical problem; however, the phenomenon is essentially due to the wave nature of the electronic states, and thus could be found in any wave phenomena. Recently, there have been several experiments on photon[8–10] and also phonon[11] localization in random media.

Let us consider a multilayer in which two types of layers A and B are arranged in a Fibonacci sequence. In order to understand the light propagation in this media, first consider an interface of two layers. (See Fig. 1.) The electric field for the light in layer A is given by

$$\mathbf{E}=\mathbf{E}_A^{(1)}\exp[i(\mathbf{k}_A^{(1)}\cdot\mathbf{x}-\omega t)] + E_A^{(2)}\exp[i(\mathbf{k}_A^{(2)}\cdot\mathbf{x}-\omega t)]. \quad (1)$$

The electric field in layer B is given by the same expression with subscript A replaced by B. We consider a polarization which is perpendicular to the plane of the light path (TE wave). The appropriate boundary condition at the interface gives

$$E_A^{(1)}+E_A^{(2)}=E_B^{(1)}+E_B^{(2)},$$
$$n_A\cos\theta_A(E_A^{(1)}-E_A^{(2)})=n_B\cos\theta_B(E_B^{(1)}-E_B^{(2)}), \quad (2)$$

where n_A and n_B are the indices of refractive of A and B, respectively, and the angles θ_A and θ_B are shown in Fig. 1. Snell's law is $\sin\theta_A/\sin\theta_B=n_B/n_A$. It is convenient to choose the two independent variables for the light as

$$E_+=E^{(1)}+E^{(2)}, \quad E_-=(E^{(1)}-E^{(2)})/i. \quad (3)$$

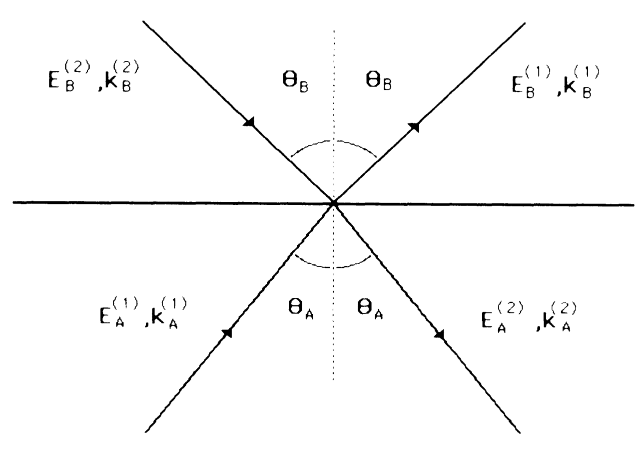

FIG. 1. Electromagnetic wave propagation across an interface of two layers.

Then (2) gives

$$\begin{bmatrix} E_+ \\ E_- \end{bmatrix}_B = T_{BA} \begin{bmatrix} E_+ \\ E_- \end{bmatrix}_A, \qquad (4)$$

where T_{BA} is given by

$$T_{BA} = \begin{bmatrix} 1 & 0 \\ 0 & n_A \cos\theta_A / n_B \cos\theta_B \end{bmatrix}. \qquad (5)$$

Also we define

$$T_{AB} = T_{BA}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & n_B \cos\theta_B / n_A \cos\theta_A \end{bmatrix}. \qquad (6)$$

The matrices T_{BA} and T_{AB} represent the light propagation across interfaces $B \leftarrow A$ and $A \leftarrow B$, respectively. The propagation within one layer is represented by

$$T_A = \begin{bmatrix} \cos\delta_A & -\sin\delta_A \\ \sin\delta_A & \cos\delta_A \end{bmatrix}, \qquad (7)$$

for a layer of type A, and the same expression for T_B in which δ_A is replaced by δ_B. The phases are given by

$$\delta_A = n_A k d_A / \cos\theta_A$$

and

$$\delta_B = n_B k d_B / \cos\theta_B, \qquad (8)$$

where k is the wave number in vacuum, and d_A and d_B are the thicknesses of the layers.

Now we are ready to consider the light propagation through a Fibonacci multilayer S_j which is sandwiched by two media of material of type A. There are F_j layers in S_j, where F_j is a Fibonacci number given recursively as $F_{j+1} = F_j + F_{j-1}$, for $j \geq 1$, with $F_0 = F_1 = 1$. For one layer A and two layers BA, the light propagations are respectively given by

$$M_1 = T_A, \quad M_2 = T_{AB} T_B T_{BA} T_A. \qquad (9)$$

It can be shown that for F_j layers, i.e., S_j, the corresponding matrix M_j is calculated as

$$M_j = M_{j-2} M_{j-1}, \qquad (10)$$

with an initial condition (9). This equation is the same as the renormalization-group equation for a quasiperiodic Schrödinger equation[3,4] and has been extensively studied.[5,12-14] It can be considered as a dynamical map and possesses a constant of motion

$$I = x_{j+1}^2 + x_j^2 + x_{j-1}^2 - 2 x_{j+1} x_j x_{j-1} - 1, \qquad (11)$$

where $x_j = \frac{1}{2} \mathrm{Tr} M_j$. This constant of motion is always positive and represents the strength of the effect of quasiperiodicity. From (9) and (10), I is explicitly written as

$$I = \tfrac{1}{4} \sin^2\delta_A \sin^2\delta_B \left(\frac{n_A \cos\theta_A}{n_B \cos\theta_B} - \frac{n_B \cos\theta_B}{n_A \cos\theta_A} \right)^2. \qquad (12)$$

For the case $n_A = n_B$ there is no quasiperiodicity and one has $I = 0$ as expected. The transmission coefficient T is given in terms of the matrix M_j as

$$T = 4/(|M_j|^2 + 2), \qquad (13)$$

where $|M_j|^2$ is the sum of the squares of the four elements of M_j. This is a quantity measured experimentally and has a rich structure with respect to a variation of either the wavelength of the light or the number of layers.

Let us consider the simplest experimental setting. Take the incident light to be normal, (i.e., $\theta_A = \theta_B = 0$) and also choose the thickness of the layers to give $\delta_A = \delta_B = \delta$ (i.e., $n_A d_A = n_B d_B$). For $\delta = m\pi$ ($\frac{1}{2}$ wavelength layer) we have $I = 0$ and the transmission is perfect. For $\delta = (m + \frac{1}{2})\pi$ ($\frac{1}{4}$ wavelength layer), I is maximum and the quasiperiodicity is most effective. (See Fig. 2.)

In addition, the $\delta = (m + \frac{1}{2})\pi$ case has the very special feature that the map (10) has a six-cycle, namely, $M_j = M_{j+6}$ for any j.[15] This implies that the transmission coefficient T has scaling about $\delta = (m + \frac{1}{2})\pi$. This is exemplified in Fig. 3 in which T is plotted against δ about $\frac{3}{2}\pi$ for S_{12} (233 layers). This is similar to the lower plot of Fig. 2 for S_9 (55 layers). Note the scale change of δ in the two figures.

In order to understand this scaling first we mention

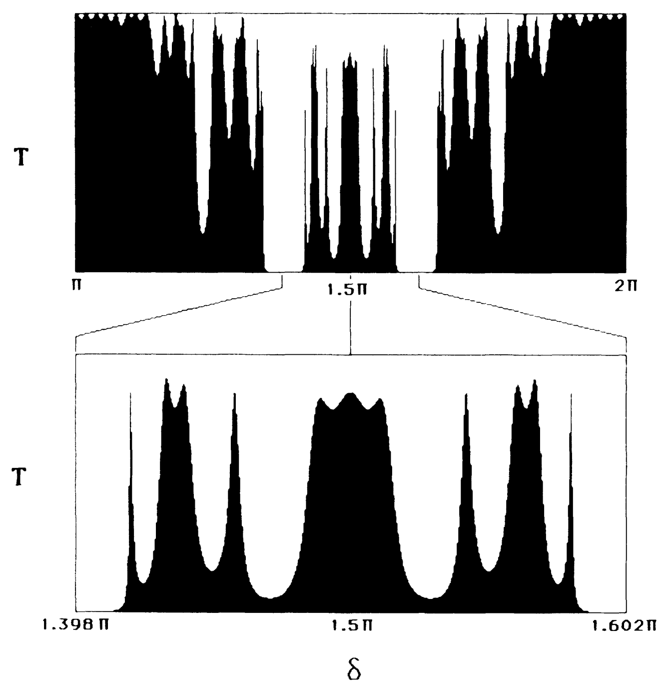

FIG. 2. The transmission coefficient T vs the optical phase length of a layer δ for a Fibonacci multilayer S_9 (55 layers). The indices of refraction are chosen as $n_A = 2$ and $n_B = 3$.

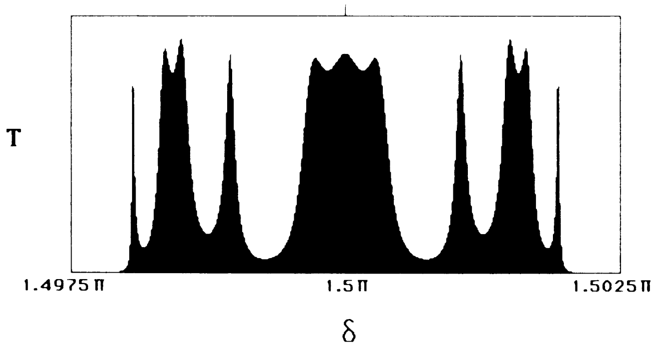

FIG. 3. The same as Fig. 2 for S_{12} (233 layers). Note the difference of the scale of δ from Fig. 2.

that the (quasi)resonance condition for a wavelength is that $x_j = \frac{1}{2}\mathrm{Tr}M_j$ is bounded. If x_j is not bounded, it grows as $x_j \sim \exp\phi^j$, where $\phi = (\sqrt{5}+1)/2$ is the golden mean. This leads to an exponential growth of R/T [R is the reflection coefficient $(1-T)$] as a function of the number of layers N. (Note that N for S_j is given by the Fibonacci number F_j which grows as ϕ^j). In fact, the trace x_j obeys the recursion relation,[3]

$$x_{j+1} = 2x_j x_{j-1} - x_{j-2}, \tag{14}$$

with an initial condition,

$$x_0 = x_1 = \cos\delta,$$

$$x_2 = \cos^2\delta[(n_A/n_B + n_B/n_A)/2]\sin^2\delta.$$

Thus, in order to locate the resonant wavelengths, one looks for δ such that the corresponding initial condition (15) gives a bounded orbit of the map (14). Apparently we have a six-cycle for the map (14) at $\delta = (m+\frac{1}{2})\pi$, since it is a subdynamical map of (10) which we know to have a six-cycle. Hence $\delta = (m+\frac{1}{2})\pi$ satisfies the resonance condition. The behavior of the orbits around the six-cycle is represented by a linearized equation, which determines the scaling behavior of the transmission coefficient. The eigenvalue of the linearized equation gives the scale factor, which is exactly calculated[12] as $[1+4(1+I)^2]^{1/2}+2(1+I)$. This gives the scale change of δ between the lower plot of Fig. 2 for S_9 (55 layers) and Fig. 3 for S_{12} (233 layers).

For the resonant case, the growth of R/T is bounded by a power of N. For the $\delta = (m+\frac{1}{2})\pi$ case the exponent is exactly given by $2|\ln(n_A/n_B)|/(3\ln\phi)$. This result is obtained from the analysis of the six-cycle of the full dynamical map (10). In addition to the power-law growth, R/T corresponding to a cycle of x_j has scaling properties and also fluctuates as N is varied. The fluctuation grows as N, and so the comparison to the universal conductance fluctuation[16] of disordered systems is an interesting problem. More details of the behavior of R/T will be published elsewhere.[8]

The resonance condition gets harder to satisfy as the number of layer N is increased, and eventually it is not satisfied for almost all δ as $N \rightarrow \infty$. However, the resonance points do exist and form a Cantor set with Lebesgue measure 0. The transmission coefficient T as a function of δ becomes singular as $N \rightarrow \infty$ and, in fact, it is a multifractal.[17] There are infinitely many scalings in which only the most prominent one at $\delta = (m+\frac{1}{2})\pi$ is discussed here.

In summary, an optical experiment with a Fibonacci multilayer is proposed. The quasilocalization (critical state) of the electromagnetic wave can be verified experimentally through the multifractal nature of the transmission coefficient.

We acknowledge useful discussions with P. C. Taylor. This work was in part supported by the U.S. Office of Naval Research, Department of the Navy under Contracts No. N00014-86-K-0258 and No. N00014-86-K-0710.

[1]For a recent review, see P. A. Lee and T. V. Ramakrishnan, Rev. Mod. Phys. **57**, 287 (1985).

[2]B. Simon, Adv. Appl. Math. **3**, 463 (1982); J. B. Sokoloff, Phys. Rep. **126**, 189 (1985).

[3]M. Kohmoto, L. P. Kadanoff, and C. Tang, Phys. Rev. Lett. **50**, 1870 (1983).

[4]S. Ostlund, R. Pandit, D. Rand, H. J. Schellnhuber, and E. D. Siggia, Phys. Rev. Lett. **50**, 1873 (1983).

[5]M. Kohmoto, B. Sutherland, and C. Tang, Phys. Rev. B **35**, 1020 (1987).

[6]B. Sutherland and M. Kohmoto, Phys. Rev. B (to be published).

[7]R. Merlin, K. Bajema, R. Clarke, F.-Y. Juang, and P. K. Bjattacharya, Phys. Rev. Lett. **55**, 1768 (1985).

[8]Y. Kuga and A. Ishimaru, J. Opt. Soc. Am. **A1**, 831 (1984); L. Tsang and A. Ishimaru, J. Opt. Soc. Am. **A2**, 2187 (1985).

[9]M. P. van Albada and A. Lagendijk, Phys. Rev. Lett. **55**, 2692 (1985); M. P. van Albada, M. P. van der Mark, and A. Lagendijk, Phys. Rev. Lett. **58**, 361 (1987).

[10]P. E. Wolf and G. Maret, Phys. Rev. Lett. **55**, 2696 (1985).

[11]S. He and J. D. Maynard, Phys. Rev. Lett. **57**, 3171 (1986).

[12]M. Kohmoto and Y. Oono, Phys. Lett. **102A**, 145 (1984).

[13]L. P. Kadanoff, to be published.

[14]M. Casdagli, Commun. Math. Phys. **107**, 295 (1986).

[15]M. Kohmoto and J. R. Banavar, Phys. Rev. B **34**, 563 (1986).

[16]A. D. Stone, Phys. Rev. Lett. **54**, 2692 (1985); P. A. Lee and A. D. Stone, Phys. Rev. Lett. **55**, 1622 (1985); P. A. Lee, A. D. Stone, and H. Fukuyama, Phys. Rev. B **35**, 1039 (1987).

[17]L. P. Kadanoff, private communication.