Exact Determination of the Percolation Hull Exponent in Two Dimensions

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By mapping the two-dimensional percolation problem on a Coulomb gas, we obtain the exact fractal dimension of the external perimeter (or "hull") of the infinite percolation cluster: $D_H = \frac{7}{4}$, in agreement with numerical estimates and a recent conjecture. We also determine an infinite set of exact exponents associated with various topologies of this hull. We argue finally that the different fractal dimensions observed recently by Grossman and Aharony, who modified the definition of the hull, are all equal to $D_e = \frac{4}{3}$.

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The percolation model, which is defined by occupation in a random way of a fraction p of bonds (or sites) of a regular lattice, has been a subject of constant interest.[1] An infinite cluster appears above the threshold p_c, corresponding to a second-order phase transition. In two dimensions, some basic features of this transition are now known exactly[2-4]; the exponent ν which describes the divergence of the correlation length $\xi \sim |p-p_c|^{-\nu}$ is[2] $\nu = \frac{4}{3}$ while the exponent γ giving the mean cluster size $\langle s \rangle \sim |p-p_c|^{-\gamma}$ is[2] $\gamma = \frac{43}{18}$. Many questions, however, remain open[5]; in particular, a lot of work has been devoted to the study of "hulls" or external cluster perimeters. Numerical studies[6-13] have given for the fractal dimension of the hull of the infinite cluster at p_c similar values, that can be represented as $D_H = 1.75 \pm 0.02$. The conjecture that D_H is related to the exponent ν by

$$D_H = 1 + 1/\nu = \tfrac{7}{4} \tag{1}$$

has been recently proposed by Sapoval, Rosso, and Gouyet.[10] Theoretical arguments for (1) have been given by Bunde and Gouyet,[14] but they rely on assumptions concerning the equivalence of hulls to diffusion fronts. Quite recently, Ziff,[15] accepting (1), derived from it other scaling exponents for perimeters and made numerical tests of them. Here we establish (1) by two independent methods. We use the formulation of percolation as a $q=1$ Potts model,[15] and by a Coulomb-gas–mapping technique[2-4] we derive a series of exact critical exponents corresponding to D_H and to other geometrical properties of the perimeters. We also show that these exponents can be extracted from the Coulomb-gas mapping of the O(n) model, for $n=1$, and in the low-temperature phase. Equation (1) now acquires the same degree of reliability as the values of ν and γ given above.

Consider first the bond problem on the square lattice. We define the *hull of a cluster* Γ *as the set of empty bonds that touch* Γ *and can be linked to infinity by a path (not restricted to the lattice) without crossing* Γ.

Bond percolation can be obtained[16] by analytic continuation of the q-state Potts model to the value $q=1$. Starting with the Hamiltonian $\beta H = -\beta \sum_{\langle ij \rangle} \delta_{\sigma_i \sigma_j}$ where $\sigma_i = 1, \ldots, q$ (integer) and $\langle i,j \rangle$ denotes nearest neighbors, one can write the high-temperature expansion of the partition function,[17]

$$Z_q = \sum_{\{\sigma\}} e^{-\beta H} = \sum_{\mathcal{G}} W(\mathcal{G}) = \sum_{\mathcal{G}} (e^{\beta}-1)^B q^C, \tag{2}$$

$W(\mathcal{G})$ being the weight of a graph $\mathcal{G}$ made of a total number B of bonds, and C connected components, i.e., clusters, including isolated points (Fig. 1). This expression now defines a model for any real q; if $q \to 1$ one recovers bond percolation with occupancy probability $p = 1 - e^{-\beta}$. For $q \in [0,4]$, there is a second-order phase transition,[17] which can be studied with use of a Coulomb-gas mapping.[3,4,18] First, a graph $\mathcal{G}$ on the original lattice $\mathcal{L}$ in the Potts model can be associated with a pol-

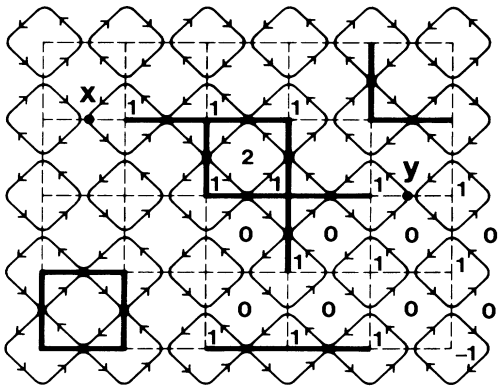

FIG. 1. A typical graph $\mathcal{G}$ in the high-temperature expansion (2) with $B=15$ bonds, $C=17$ clusters, $S=30$ sites, and $L=2$ internal loops. $\mathcal{G}$ is in one-to-one correspondence with a polygon decomposition (here $P=19$) of the surrounding lattice. After an arbitrary orientation the polygons become walls between regions of constant height in a SOS model. Height variables θ are located on the sites of $\mathcal{L}$ and its dual $\mathcal{D}$. Some values (in $\pi/2$ units) are indicated on the figure in accordance with the polygon orientations. X,Y belonging to the same polygon $\mathcal{P}$, this graph contributes also to $G_1(X-Y)$.

ygon decomposition[19] of the surrounding lattice $\mathcal{S}$, here another square lattice, the sites of which are the midpoints of the edges of $\mathcal{L}$ (Fig. 1). The rule is that some vertices of $\mathcal{S}$ are cut open to let the bonds of $\mathcal{L}$ go through unintersected. This also applies to the edges of the dual lattice $\mathcal{D}$ of $\mathcal{L}$. For a lattice $\mathcal{L}$ with a total number of sites S, one has (Euler's relation) $L=B+C-S$, where L is the number of loops within the clusters of the graph $\mathcal{G}$. On $\mathcal{S}$, the total number of polygons one can draw around each cluster of $\mathcal{G}$ and in each loop reads $P=L+C$. Hence $Z_q(2)$ can be rewritten as

$$Z_q=q^{S/2}\sum_{\mathcal{G}}[(e^{\beta}-1)q^{-1/2}]^B q^{P/2}. \tag{3}$$

In this form one can now formulate Z_q as the partition function of a solid-on-solid (SOS) model.[2,4] First we orient arbitrarily each polygon (these orientations will be summed over at the end). The polygons can then be considered as walls between regions of constant heights, the height variables θ being located on the sites of $\mathcal{L}$ and of its dual $\mathcal{D}$ (Fig. 1). By convention, the associated θ difference is taken to be $\pi/2$, the highest θ being on the left of each arrow. The Boltzmann weight W of a given SOS configuration $\mathcal{C}$ is obtained[3,4] as a product of phase factors e^{iu} (e^{-iu}) associated with each left (right) corner of a wall, and a factor $(e^{\beta}-1)q^{-1/2}$ for each bond of the original lattice $\mathcal{L}$ which belongs to a region of constant height. Since the difference between the total number of left and right turns for a polygon on the square lattice is ± 4 and since the orientations in the SOS model are summed over independently, resulting in a factor $2\cos 4u$ for each polygon, one has[3,4] $Z_q=q^{S/2}Z_{\mathrm{SOS}}$ provided that $q^{1/2}=2\cos 4u$. The critical point β_c of (2) or (3) is obtained by duality[17]: $(\exp\beta_c-1)q^{-1/2}=1$. At β_c, the SOS model renormalizes onto the vacuum phase of the Coulomb gas with a coupling constant g given by[2-4]

$$q=2+2\cos(\pi g/2),\quad g\in[2,4], \tag{4}$$

with $8u/\pi=|2-g/2|$ mod4. The main exponents of the Potts model can then be derived exactly,[2-4,18] and for percolation one finds[2,3] $\nu=\frac{4}{3}$, $\gamma=\frac{43}{18}$. (These results, as well as those presented hereafter, can be considered as exact, provided that[3,4] there is no "intermediate fixed point" in the renormalization flow diagram.)

To determine the exact hull exponents, we now introduce a Potts correlation function

$$G_1(X-Y)=\frac{1}{Z_q}\sum_{\mathcal{G}_1}W(\mathcal{G}_1), \tag{5}$$

the sum being taken over all graphs $\mathcal{G}_1$ where X,Y are two points on the surrounding lattice at the corners of the same polygon $\mathcal{P}$ (Fig. 1). For calculating (5) we modify the orientation of one line of $\mathcal{P}$ so that both lines of $\mathcal{P}$ go only from X to Y. In a SOS language, the resulting configuration $\mathcal{C}_1'$ then describes[2-4] a dislocation with a vortex at X and an antivortex at Y: Along a closed path around X (Y) the height varies by π ($-\pi$). These vortices correspond,[2-4] in the Coulomb-gas picture, to magnetic charges $m_X=-m_Y=\frac{1}{2}$. The Boltzmann weight $W(\mathcal{C}_1')$ in the SOS model does not, however, exactly correspond to (5). X,Y being fixed, there is a new curvature factor[3,4] e^{4iu} (e^{-4iu}) for each left (right) turn of the polygon around one extremity. This can be compensated by the multiplication of $W(\mathcal{C}_1')$ by a spin-wave factor $\exp[i(e_X\theta_X+e_Y\theta_Y)]$ where e_X,e_Y are two electric charges given by $e_X=e_Y=-8u/\pi$:

$$G_1(X-Y)\propto\frac{1}{Z_q}\sum_{\mathcal{C}_1'}W(\mathcal{C}_1')e^{i(e_X\theta_X+e_Y\theta_Y)}. \tag{6}$$

G_1 appears thus as the correlation function of two combinations of a vortex and a spin wave with magnetic and (dominant) electric charges $(\frac{1}{2},\frac{1}{2}g-2)$ and $(-\frac{1}{2},\frac{1}{2}g-2)$. It decays at criticality as $|X-Y|^{-2x_1}$ with

$$x_1=-gm_Xm_Y/2-e_Xe_Y/2g, \tag{7}$$

or

$$x_1=g/8-(4-g)^2/8g. \tag{8}$$

This is valid for any value of q in the Potts model. For percolation, $q=1$, and by (4) $g=\frac{8}{3}$. Thus we find the key result $x_1=\frac{1}{4}$. For p close to p_c (equal to $\frac{1}{2}$ on the square lattice), G_1 has the scaling form

$$G_1(X-Y)\simeq|X-Y|^{-2x_1}F_1\left(\frac{|X-Y|}{|p-p_c|^{-\nu}}\right). \tag{9}$$

Integration over Y gives then the mean length of a polygon $\mathcal{P}$ (Fig. 1) which diverges as

$$\langle l(\mathcal{P})\rangle\sim|p-p_c|^{-\gamma_1}, \tag{10}$$

where $\gamma_1=(2-2x_1)\nu=2$. Equation (10) describes also the divergence of the mean number of bonds in a hull. This agrees with numerical calculations of Ref. 8 and gives $\gamma_1=2.0\pm0.1$ (γ_1 is also denoted by[9] ψ and[15] γ'). Using standard scaling relations generalized to perimeters,[15] one gets the exponent β_1 with which the probability P_1 that a point belongs to the hull of the infinite cluster grows for $p\geq p_c$,

$$P_1\sim(p-p_c)^{\beta_1}, \tag{11}$$

where $\beta_1=\nu-\frac{1}{2}\gamma_1=\nu x_1=\frac{1}{3}$. Finally γ_1 can be related[9,15] to the fractal dimension D_H by $\gamma_1=2\nu(D_H-1)$; thus

$$D_H=2-x_1=\tfrac{7}{4}. \tag{12}$$

We can now generalize[20,21] this result. We introduce correlation functions

$$G_k(X-Y)=\frac{1}{Z_q}\sum_{\mathcal{G}_k}W(\mathcal{G}_k), \tag{13}$$

where the $\mathcal{G}_k$ are graphs formed by k polygons which

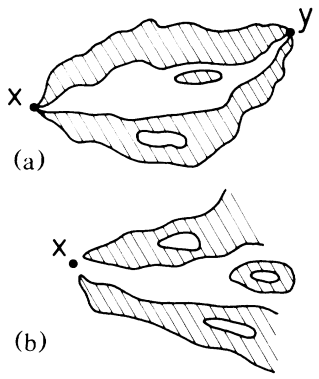

FIG. 2. (a) A cluster made of k ($=2$) connected occupied bands, pinched together at X and Y. (b) A point X of the infinite-cluster perimeter, where k ($=2$) large occupied bands (hatched peninsulas) of the infinite cluster come close together.

join a fixed neighborhood of X to a fixed neighborhood of Y. Modifying the orientation of part of these polygons in such a way that one can circulate only from X to Y, one gets in the SOS model a dislocation with a vortex of magnetic charge $m_X = k/2$ at X and $m_Y = -k/2$ at Y. The weights in (13) impose in the SOS model the same additional phase factor $\exp[(-8u/\pi)i(\theta_X+\theta_Y)]$. G_k decays thus at criticality like $|X-Y|^{-2x_k}$ with [see Eq. (7)]

$$x_k = \tfrac{1}{2} g(\tfrac{1}{2}k)^2 - (4-g)^2/8g. \tag{14}$$

For percolation ($g = \frac{8}{3}$) we have

$$x_k = (4k^2-1)/12. \tag{15}$$

The physical interpretation of the exponents is as follows. Like in Eq. (9), G_k reads for p near p_c

$$G_k(X-Y) \simeq |X-Y|^{-2x_k} F_k\left(\frac{|X-Y|}{|p-p_c|^{-\nu}}\right). \tag{16}$$

Integrating with respect to Y gives the singular part of the mean number of clusters, the external perimeter of which has the special topology of k bands pinched at their extremities [Fig. 2(a)], $\langle n_k\rangle \sim |p-p_c|^{-\gamma_k}$ with $\gamma_k = (2-2x_k)\nu$. One can also consider the probability P_k that a point belongs to a region of the perimeter of the infinite cluster, where the latter has the special topology of k bands ("peninsulas") coming close together [Fig. 2(b)]. It grows as $P_k \sim (p-p_c)^{\beta_k}$, where $\beta_k = \nu - \frac{1}{2}\gamma_k = \nu x_k$. For $k=2$, it is interesting to note that two touching peninsulas are equivalent to a *cutting bond*.[5,22] We find $x_2 = \frac{5}{4}$; hence we obtain a fractal dimension $D_{\text{red}} = 2-\beta_2/\nu = \frac{3}{4} = 1/\nu$, thus giving another determination of a well-known result.[22]

It is worth noting that the basic exponents in (15) and (16) can also be derived from the O(n) model. Consider *site* percolation on a triangular lattice (Fig. 3). Polygons on the dual hexagonal lattice $\mathcal{H}$ separate occupied from empty sites. The hull configuration so obtained (Fig. 3) is clearly in one-to-one correspondence to a gas of loops on $\mathcal{H}$. This is just the O(n) model introduced by Nienhuis[3] with a partition function

$$Z_n = \sum_{\mathcal{G}} \beta^B n^P, \tag{17}$$

the sum being taken over graphs formed by P nonintersecting self-avoiding loops of total length B on $\mathcal{H}$. At the threshold $p_c = \frac{1}{2}$, all polygons for percolation have the same weight, 1, which corresponds to Eq. (17) for $n=1$, $\beta=1$. The critical point of the O(n) model on $\mathcal{H}$ is exactly known,[3] and for $n=1$, $\beta_c = 1/\sqrt{3}$. We are thus in the *low*-temperature phase, which is known to be also critical, renormalizing onto the Coulomb gas with coupling such that[3,4] $n = -2\cos\pi g'$ and $g' \in [0,1]$. Hence for $n=1$, $g' = \frac{2}{3}$. Then the correlation functions $G_k(X-Y)$ [Eq. (13)] correspond now in the O(n) model to correlation functions of products of $2k$ spins at X and Y. The critical exponents x'_{2k} with which these functions decay are known[4,21] by mapping of the O(n) model onto a Coulomb gas and read

$$x'_{2k} = \tfrac{1}{2} g' k^2 - (1/2g')(g'-1)^2. \tag{18}$$

For $g' = \frac{2}{3}$, Eq. (18) coincides with (15) $x'_{2k} \equiv x_k$, as expected. More generally, we state that the q-state Potts model at its critical point (for $0 \le q \le 4$) can be related to an O(n) model in its low-temperature phase with $n = q^{1/2}$ ($0 \le n \le 2$).

Let us comment finally on the generality of our results. We expect $D_H = \frac{7}{4}$ to be universal provided one defines the hull of a cluster Γ in a "natural" way. For *bond* percolation, the mapping of the Potts model explained above is the same for all lattices[3,4] and the hull has always the same definition. The situation is more complicated for *site* percolation. Recently Grossmann and Aharony[23] have numerically observed that when defining the hull on the square lattice as the set of empty sites which touch Γ and are related to infinity by a path of *nearest* or *next nearest* neighbor connections avoiding

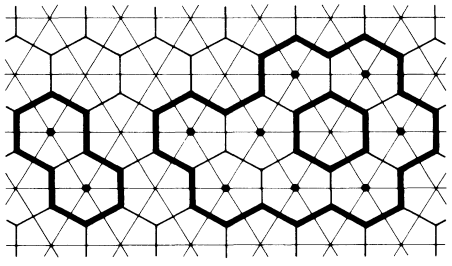

FIG. 3. Site percolation on the triangular lattice. Full circles represent occupied sites. The hull perimeter (bold lines) is drawn on the dual hexagonal lattice.

Γ, one gets $D_H \simeq \frac{7}{4}$, while the restriction to *nearest* neighbor connections gives a different result, $D_e = 1.37 \pm 0.03$. Similarly, Meakin and Family[24] have found $D_e = 1.343 \pm 0.002$. Now, Coniglio *et al.*[25] have indicated that the percolation hull (defined in a natural way as in this Letter; in particular see our Fig. 3) with $D_H = \frac{7}{4}$ is identical to a self-avoiding walk (SAW) at the Θ point,[26] which gives $\nu_\Theta = 1/D_H = \frac{4}{7}$, in good agreement with numerical estimates.[27] This Θ point is an unstable tricritical point.[26,28] Hence, any macroscopic restriction on the hull configurations like in Refs. 23 and 24, being equivalent to further repulsive interactions between bonds on the hull SAW, automatically drive the latter to its excluded-volume fixed point. The fractal dimension is then well known,[3] $D = \nu_{\mathrm{SAW}}^{-1} = \frac{4}{3}$. We thus suggest that D_e,[23,24] and also the whole hierarchy of dimensions introduced in Ref. 23, are all equal to $\frac{4}{3}$, in excellent agreement with the numerical results. We hope[29] also to extend the analogy of Ref. 25 for determining from the hull exponents given here the polymer exponents at the Θ point.

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