Field Correlations within a Fluctuating Homogeneous Medium

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(Received 7 August 1986)

The general form of the field correlations within a fluctuating homogeneous medium is determined in terms of the corresponding source correlations. Under very general conditions, the degree of spectral coherence has a universal form, independent of the source, if the fluctuations are isotropic. If they are anisotropic, the degree of coherence at separations that are large compared with the wavelength directly yields the generalized dynamical structure factor of the medium.

PACS numbers: 42.10.Mg, 05.40.+j, 42.20.Ee

The space-time correlation function of a fluctuating physical quantity, represented by the associated complex analytic signal[1] $Q(\mathbf{r},t)$, is defined by

$$\Gamma_Q(\mathbf{r}_1,\mathbf{r}_2,t_1,t_2)=\langle Q^*(\mathbf{r}_1,t_1)Q(\mathbf{r}_2,t_2)\rangle, \tag{1}$$

where the angular brackets denote the ensemble average. The function Γ_Q (where Q could stand for[2] density, current, polarization, . . .) plays a fundamental role in the physics of many-particle systems.[3,4]

We assume that $Q(\mathbf{r},t)$ plays the role of source density[5] for a wave field, represented by the complex analytic signal $V(\mathbf{r},t)$ (V could stand for[2] sound waves, electromagnetic waves, . . .). Thus, for the Fourier components[6] $Q(\mathbf{r},\omega)$ and $V(\mathbf{r},\omega)$,

$$(\Delta+k^2)V(\mathbf{r},\omega)=-4\pi Q(\mathbf{r},\omega),$$

where $k=\omega/v$ (v is the velocity of propagation at ω). The space-time correlation function for the fluctuations of V associated with (1) is

$$\Gamma_V(\mathbf{r}_1,\mathbf{r}_2,t_1,t_2)=\langle V^*(\mathbf{r}_1,t_1)V(\mathbf{r}_2,t_2)\rangle. \tag{2}$$

For stationary random processes, Γ_Q depends on t_1,t_2 only through $t=t_2-t_1$, which entails the same property for Γ_V. It then follows from a generalization of the Wiener-Khintchine theorem that

$$\Gamma_A(\mathbf{r}_1,\mathbf{r}_2,t_1,t_1+t)=\int_0^\infty W_A(\mathbf{r}_1,\mathbf{r}_2,\omega)e^{-i\omega t}d\omega,$$

where A stands for either Q or V, and $W_A(\mathbf{r}_1,\mathbf{r}_2,\omega)$ is called the cross-spectral density of A. In particular, $W_A(\mathbf{r},\mathbf{r},\omega)\geq 0$ ($A=Q,V$) is the spectral density of A at $\mathbf{r}$.

The relation between W_V and W_Q is[7]

$$W_V(\mathbf{r}_1,\mathbf{r}_2,\omega)=\lim_{\varepsilon\to 0^+}W_V(\mathbf{r}_1,\mathbf{r}_2,\omega,\varepsilon), \tag{3}$$

where $\varepsilon\to 0^+$ denotes the limit from above,

$$W_V(\mathbf{r}_1,\mathbf{r}_2,\omega,\varepsilon)=\int\int G^{(+)*}(k+i\varepsilon,|\mathbf{r}_1-\mathbf{r}_1'|)W_Q(\mathbf{r}_1',\mathbf{r}_2',\omega)G^{(+)}(k+i\varepsilon,|\mathbf{r}_2-\mathbf{r}_2'|)d^3r_1'\,d^3r_2', \tag{4}$$

$G^{(+)}(k,r)=\exp(ikr)/r$, and the space integrals in (4) are extended over the source domain. Here we deal with an unbounded medium,[8] so that they are extended throughout all space. This justifies the ε-limiting procedure in (3): An arbitrarily small $\varepsilon>0$ in an infinite medium gives rise to absorption, which always exists in an actual medium.[9]

We want to investigate the consequences of *statistical homogeneity;* then W_Q depends on $\mathbf{r}_1$ and $\mathbf{r}_2$ only through $\mathbf{r}=\mathbf{r}_2-\mathbf{r}_1$. For this purpose we make the change of variables $\mathbf{R}=(\mathbf{r}_1+\mathbf{r}_2)/2$, $\mathbf{r}=\mathbf{r}_2-\mathbf{r}_1$, with similar definitions for the primed coordinates in (4). If we perform this change of variables, the integration over $\mathbf{R}'$ reduces to the basic integral

$$I=\int\frac{\exp[ik(|\mathbf{R}'+\mathbf{\Delta}|-|\mathbf{R}'-\mathbf{\Delta}|)-\varepsilon(|\mathbf{R}'+\mathbf{\Delta})|+|\mathbf{R}'-\mathbf{\Delta}|)]}{|\mathbf{R}'+\mathbf{\Delta}|\,|\mathbf{R}'-\mathbf{\Delta}|}d^3R',$$

where $\mathbf{\Delta}=(\mathbf{r}-\mathbf{r}')/2$. This integral can be computed by going over to spheroidal coordinates,[10] with the following result:

$$I=\pi\sin(2k|\mathbf{\Delta}|)\exp(-2\varepsilon|\mathbf{\Delta}|)/(\varepsilon k|\mathbf{\Delta}|).$$

It then follows that W_V also depends only on $\mathbf{r}=\mathbf{r}_2-\mathbf{r}_1$, and is given by

$$W_V(\mathbf{r},\omega,\varepsilon)=(2\pi/\varepsilon)\int j_0(k|\mathbf{r}-\mathbf{r}'|)\exp(-\varepsilon|\mathbf{r}-\mathbf{r}'|)W_Q(\mathbf{r}',\omega)\,d^3r', \tag{5}$$

where $j_0(kr)=\sin(kr)/kr$ is the spherical Bessel function of order zero.

The complex degree of spectral coherence[11] for an arbitrary homogeneous medium is therefore given by

$$\mu_V(\mathbf{r},\omega)=\lim_{\varepsilon\to 0^+}\frac{\int j_0(k|\mathbf{r}-\mathbf{r}'|)\exp(-\varepsilon|\mathbf{r}-\mathbf{r}'|)W_Q(\mathbf{r}',\omega)d^3r'}{\int j_0(kr')\exp(-\varepsilon r')W_Q(\mathbf{r}',\omega)d^3r'}, \tag{6}$$

where $r'=|\mathbf{r}'|$. Note that the factor $1/\varepsilon$ in (5) canceled out in the ratio in (6).

Let us reexpress the results in terms of the spatial Fourier components $\tilde{W}_Q(\mathbf{K},\omega)$ of $W_Q(\mathbf{r},\omega)$,

$$W_Q(\mathbf{r},\omega)=\int \tilde{W}_Q(\mathbf{K},\omega)\exp(i\mathbf{K}\cdot\mathbf{r})d^3K. \tag{7}$$

The (nonnegative) quantity $\tilde{W}_Q(\mathbf{K},\omega)$ is the "double" spectral density of $Q(r,t)$. [For particle-density fluctuations, $\tilde{W}_Q(\mathbf{K},\omega)$ is the *dynamical structure factor.*[3,4]] Substituting (7) into (6), we find that

$$\mu_V(\mathbf{r},\omega)=\lim_{\varepsilon\to 0^+}\frac{\int[f(K-k,\varepsilon)-f(K+k,\varepsilon)]\tilde{W}_Q(\mathbf{K},\omega)\exp(i\mathbf{K}\cdot\mathbf{r})K^{-1}d^3K}{\int[f(K-k,\varepsilon)-f(K+k,\varepsilon)]\tilde{W}_Q(\mathbf{K},\omega)K^{-1}d^3K}, \tag{8}$$

where $K=|\mathbf{K}|$ and $f(u,\varepsilon)=\varepsilon/(u^2+\varepsilon^2)$. Note that $f(u,\varepsilon)\to\pi\delta(u)$ as $\varepsilon\to 0^+$. *The equivalent results (6) and (8) hold for an arbitrary homogeneous medium.*

From now on, we assume that

$$\int j_0(kr')W_Q(\mathbf{r}',\omega)d^3r'\neq 0, \tag{9}$$

or, equivalently, in terms of the Fourier transform,

$$\int\delta(K-k)\tilde{W}_Q(\mathbf{K},\omega)K^{-1}d^3K=k\int\tilde{W}_Q(k\hat{\mathbf{s}},\omega)\,d\Omega_s\neq 0, \tag{10}$$

where $\hat{\mathbf{s}}$ denotes a unit vector, and $d\Omega_s$ is the element of solid angle around the $\hat{\mathbf{s}}$ direction.

According to (5), $W_V(\mathbf{r},\omega,\varepsilon)$ diverges as $\varepsilon\to 0^+$ when (9) holds, but, by (6) and (8), μ. remains well defined, and it is given by the equivalent expressions

$$\mu_V(\mathbf{r},\omega)=\int j_0(k|\mathbf{r}-\mathbf{r}'|)W_Q(\mathbf{r}',\omega)d^3r'\left[\int j_0(kr')W_Q(\mathbf{r}',\omega)d^3r'\right]^{-1}, \tag{11}$$

$$\mu_V(\mathbf{r},\omega)=\int\tilde{W}_Q(k\hat{\mathbf{s}},\omega)\exp(ik\mathbf{r}\cdot\hat{\mathbf{s}})d\Omega_s\left[\int\tilde{W}_Q(k\hat{\mathbf{s}},\omega)\,d\Omega_s\right]^{-1}. \tag{12}$$

We expect that *(9) [(10)] is valid in most situations, so that (11) [(12)] will hold, barring exceptional cases.*[12]

We see from (11) that μ_V is now a solution of the Helmholtz equation

$$(\Delta+k^2)\mu_V(\mathbf{r},\omega)=0, \tag{13}$$

in which all memory of the source is erased. This is a consequence of the limit $\varepsilon\to 0^+$; in fact, one can show that

$$[\Delta+(k\pm i\varepsilon)^2]\mu_V(\mathbf{r},\omega,\varepsilon)=-\frac{2\varepsilon}{N_\varepsilon}\int\frac{\exp[\mp i(k\mp i\varepsilon)|\mathbf{r}-\mathbf{r}'|]}{|\mathbf{r}-\mathbf{r}'|}W_Q(\mathbf{r}',\omega)d^3r',$$

where N_ε is the denominator of (6) and upper and lower signs are matched. It is also clear [see (10)] that only "on-shell" components ($|\mathbf{K}|=k$) contribute to (12).

If, besides homogeneity, we also have statistical isotropy, so that $W_Q(\mathbf{r},\omega)$ depends on $\mathbf{r}$ only through $r=|\mathbf{r}|$, it follows from (11) [(12)] that

$$\mu_V(\mathbf{r},\omega)=j_0(kr)=\sin kr/kr. \tag{14}$$

Thus, we obtain the remarkable result that *all homogeneous and isotropic source cross-spectral densities $W_Q(r,\omega)$ satisfying (9), regardless of how different their spatial behavior may be, lead to the same degree of spectral coherence (14) for the field.* This also follows from the fact that (14) is the unique isotropic solution of (13) such that $\mu_V(0,\omega)=1$.

Special cases of this result are known: the important case of *black-body radiation*[9] and that of the field correlations within a delta-correlated primary spherical source of radius much larger than the wavelength.[13] However, black-body radiation can also be produced by inhomogeneous sources.

Let us assume now that the source density is *anisotropic,* but that it is sufficiently dilute within the background medium that we can still treat the wave propagation as isotropic (as before); this assumption is commonly made to simplify the treatment.[14] We expand $\tilde{W}_Q(k\hat{\mathbf{s}},\omega)$ in spherical harmonics $Y_{lm}(\hat{\mathbf{s}})$:

$$\tilde{W}_Q(k\hat{\mathbf{s}},\omega)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\tilde{w}_{lm}(k)Y_{lm}(\hat{\mathbf{s}}). \tag{15}$$

In many cases, the anisotropy is not very strong, so that (15) can be cut off at $l=l_{\max}$, with $l_{\max}$ not $\gg 1$. Substituting (15) into (12), we find

$$\mu_V(\mathbf{r},\omega)=2[\sqrt{\pi}/\tilde{w}_{00}(k)]\sum_{l=0}^{\infty}\sum_{m=-l}^{l}i^l\tilde{w}_{lm}(k)j_l(kr)Y_{lm}(\hat{\mathbf{r}}), \tag{16}$$

where $\hat{\mathbf{r}}=\mathbf{r}/r$ and j_l is the spherical Bessel function of order l.

Let

$$\eta_Q(k\hat{\mathbf{s}},\omega)\equiv\frac{\tilde{W}_Q(k\hat{\mathbf{s}},\omega)}{\int\tilde{W}_Q(k\hat{\mathbf{s}},\omega)\,d\Omega_s}=\eta_Q^{(\text{even})}+\eta_Q^{(\text{odd})} \tag{17}$$

be the double spectral density normalized over the sphere $|\mathbf{K}|=k$, decomposed into its even-parity part $\eta_Q^{(\text{even})}$ and odd-parity part $\eta_Q^{(\text{odd})}$ with respect to inversion in $\mathbf{K}$ space ($\mathbf{K}\to-\mathbf{K}$); for an inversion-symmetric medium, $\eta_Q^{(\text{odd})}=0$. Then, for $kr\gg l_{\max}$, we find from (16)

$$(4\pi)^{-1}\mu_V(\mathbf{r},\omega)\approx\eta_Q^{(\text{even})}(k\hat{\mathbf{r}},\omega)\sin(kr)/kr-i\eta_Q^{(\text{odd})}(k\hat{\mathbf{r}},\omega)\cos(kr)/kr\quad(kr\gg l_{\max}). \tag{18}$$

This result gives the asymptotic behavior of μ_V for an arbitrary homogeneous medium satisfying (10). Note that (14) is a special case of (18), with $l_{\max}=0$, and that the $\cos(kr)/kr$ contribution appears only in inversion-asymmetric media.

Thus, if one can determine both the modulus and the phase[15] of $\mu_V(\mathbf{r},\omega)$ within a medium with anisotropic source density as a function of $\hat{\mathbf{r}}$, for a fixed separation $r\gg l_{\max}\lambda\!\!\!^{-}$ ($\lambda\!\!\!^{-}\equiv 1/k$), the result (18) yields $\eta_Q(k\hat{\mathbf{r}},\omega)$ and, to obtain $W_Q(k\hat{\mathbf{r}},\omega)$, it suffices to determine its magnitude in one specific direction. Note that $\mu_V(\mathbf{r},\omega)$ along $\hat{\mathbf{r}}$ determines η_Q along the *same* direction $\hat{\mathbf{r}}$ in $\mathbf{K}$ space, in contrast with the well-known light scattering technique, where the direction of $\mathbf{K}$ is determined by momentum conservation.[4]

As a specific example, let us consider a nematic liquid crystal, with average orientation aligned by a magnetic field $\mathbf{H}=H\hat{\mathbf{z}}$. We employ as a simplified model the Ornstein-Zernike Lorentzian form[14,16]

$$\tilde{W}_Q(\mathbf{K},\omega)=A(\omega^2+\Gamma^2)^{-1}[C_{\parallel}K_z^2+C_{\perp}(K_x^2+K_y^2)+\chi H^2]^{-1}, \tag{19}$$

where we take[16] A as constant. We consider only the asymptotic form of μ_V for weak anisotropy. We then find that

$$\mu_V(r,\theta,\omega)\approx(1-\alpha/3)^{-1}(1-\alpha\cos^2\theta)\sin(kr)/kr,$$

$$\alpha\equiv[(C_{\parallel}-C_{\perp})/C_{\perp}](1+\lambda\!\!\!^{-2}/\xi^2)^{-1}\quad(kr\gg 1,\ |\alpha|\ll 1),$$

where $\xi=(C_{\perp}/\chi H^2)^{1/2}$ is the magnetic coherence length.[14] The shorter the value of ξ, the less the anisotropy is felt, as would be expected.

This work was supported by the National Science Foundation, and by the Coordenação de Aperfeiçoamento de Pessoal do Ensino Superior (Brazil), the Conselho Nacional de Desenvolvimento Científico e Tecnológico (Brazil), and the Financiadora de Estudos e Projetos (Brazil).

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[1]M. Born and E. Wolf, *Principles of Optics* (Pergamon, Oxford, 1980), 6th ed., Sect. 10.2.

[2]For simplicity, we deal with scalar quantities, but the treatment can be readily extended to quantities of different tensorial character.

[3]L. van Hove, Phys. Rev. **95**, 249 (1954).

[4]D. Forster, *Hydrodynamic Fluctuations, Broken Symmetry, and Correlation Functions* (Benjamin, Reading, MA, 1975).

[5]In particular, Q could be a disturbance arising from the response to an external perturbation.

[6]When Q and V are stationary random variables, a more careful discussion is required [cf. E. Wolf, J. Opt. Soc. Am. **72**, 343 (1982), and J. Opt. Soc. Am. A **3**, 76 (1986)].

[7]E. Wolf and W. H. Carter, J. Opt. Soc. Am. **68**, 953 (1978). That paper deals with bounded sources, so that the ε-limiting procedure of (3) is not employed; also, the definitions of W_A differ from ours by complex conjugation.

[8]As usual, this means that we consider points well within an actual (bounded) medium, where boundary effects are negligible.

[9]L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media* (Addison-Wesley, Reading, MA, 1960), Sects. 88, 89.

[10]C. Zemach and A. Klein, Nuovo Cimento **10**, 1078 (1958).

[11]This is defined [cf. L. Mandel and E. Wolf, J. Opt. Soc. Am. **66**, 529 (1976)] by

$$\mu_V(\mathbf{r}_1,\mathbf{r}_2,\omega)=W_V(\mathbf{r}_1,\mathbf{r}_2,\omega)/[W_V(\mathbf{r}_1,\mathbf{r}_1,\omega)W_V(\mathbf{r}_2,\mathbf{r}_2,\omega)]^{1/2},$$

which, for a homogeneous medium, reduces to $W_V(\mathbf{r},\omega)/W_V(\mathbf{0},\omega)$.

[12]When the right-hand side of (9) [(10)] vanishes, the same happens with the numerator of (11) [(12)], so that we must go back to (6) and (8) and determine their limit. In general, μ_V will decrease faster as $|\mathbf{r}|\rightarrow\infty$ when this happens than when (9) [(10)] holds. There is some analogy between this situation and that encountered with *nonradiating sources,* for which the analogs of (9) and (10) vanish when W_Q is replaced by the source density [cf. N. J. Bleistein and J. K. Cohen, J. Math. Phys. **18**, 194 (1977); K. Kim and E. Wolf, Opt. Commun. **59**, 1 (1986)].

[13]J. T. Foley, W. H. Carter, and E. Wolf, J. Opt. Soc. Am. A **3**, 1090 (1986).

[14]P. G. de Gennes, *The Physics of Liquid Crystals* (Clarendon, Oxford, 1975), Sect. 3.4.

[15]The phase need not be determined for media with inversion symmetry.

[16]M. J. Stephen and J. P. Straley, Rev. Mod. Phys. **46**, 617 (1974). We take Γ as constant, whereas actually (Ref. 14) $\Gamma=\Gamma(\mathbf{K})$.