Potentials Which Cause the Same Scattering at all Energies in One Dimension

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Explicit scattering solutions of the one-dimensional Schrödinger equation are given. A one-parameter family of the potentials considered here causes the same scattering at all energies. The previously published explicit examples of nonuniqueness in the one-dimensional inverse quantum problem are special cases of the potentials given here.

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Consider the one-dimensional Schrödinger equation,

$$d^2\psi(k,x)/dx^2+k^2\psi(k,x)=V(x)\psi(k,x).$$

If the potential $V(x)$ vanishes as $x\to\pm\infty$ in some sense, we find two linearly independent solutions ψ_l and ψ_r, which are usually called physical solutions from the left and from the right respectively, with the boundary conditions

$$\begin{bmatrix}\psi_l(k,x)\\ \psi_r(k,x)\end{bmatrix}=\begin{bmatrix}T(k)\\ R(k)\end{bmatrix}e^{ikx}+\begin{bmatrix}0\\ 1\end{bmatrix}e^{-ikx}+o(1),\ \text{as } x\to\infty,$$

and

$$\begin{bmatrix}\psi_l(k,x)\\ \psi_r(k,x)\end{bmatrix}=\begin{bmatrix}L(k)\\ T(k)\end{bmatrix}e^{-ikx}+\begin{bmatrix}1\\ 0\end{bmatrix}e^{ikx}+o(1),\ \text{as } x\to-\infty,$$

where

$$S(k)\equiv\begin{bmatrix}T(k) & R(k)\\ L(k) & T(k)\end{bmatrix}$$

is the scattering matrix, $T(k)$ is the transmission coefficient, and $R(k)$ and $L(k)$ are the reflection coefficients from the right and from the left, respectively. Good reviews of the scattering and inverse scattering problem for the Schrödinger equation exist in the literature.[1-3]

Letting

$$m_l(k,x)\equiv[1/T(k)]e^{-ikx}\psi_l(k,x)$$

and

$$m_r(k,x)\equiv[1/T(k)]e^{ikx}\psi_r(k,x)$$

we obtain[4]

$$d^2m_l(k,x)/dx^2+2ik\,dm_l(k,x)/dx=V(x)m_l(k,x)$$

and

$$d^2m_r(k,x)/dx^2-2ik\,dm_r(k,x)/dx=V(x)m_r(k,x),$$

with the boundary conditions

$$m_l(k,x)=1+o(1)\ \text{and}\ dm_l(k,x)/dx=o(1),\ \text{as } x\to\infty,$$

$$m_r(k,x)=1+o(1)\ \text{and}\ dm_r(k,x)/dx=o(1),\ \text{as } x\to-\infty.$$

If we let

$$m_l(k,x)\equiv\sum_{n=0}^{\infty}\left(\frac{i}{k}\right)^n f_n(x)\ \text{and}\ m_r(k,x)\equiv\sum_{n=0}^{\infty}\left(\frac{-i}{k}\right)^n g_n(x),$$

we obtain[5]

$$f_0(x)=1;\; f_n(x)=\frac{1}{2}\frac{d}{dx}f_{n-1}(x)+\frac{1}{2}\int_x^\infty dy\, V(y)f_{n-1}(y),\quad n\geq 1; \tag{1}$$

and

$$g_0(x)=1;\; g_n(x)=\frac{1}{2}\frac{d}{dx}g_{n-1}(x)-\frac{1}{2}\int_{-\infty}^x dy\, V(y)g_{n-1}(y),\quad n\geq 1. \tag{2}$$

Consider the family of potentials $V(x,\alpha,\beta,c,M,N)$ defined as

$$V(x,\alpha,\beta,c,M,N)=c\delta(x)-2\theta(x)[P'(x,\alpha,N)/P(x,\alpha,N)]'-2\theta(-x)[Q'(x,\beta,M)/Q(x,\beta,M)]',$$

where α, β, and c are real parameters, M and N are positive integers, $\delta(x)$ is the Dirac delta function, $\theta(x)$ is the Heaviside step function, the prime denotes the x derivative, and

$$P(x,\alpha,N)\equiv(x+1)^{N(N+1)/2}+\alpha(x+1)^{(N-2)(N-1)/2}, \tag{3}$$

and

$$Q(x,\beta,M)\equiv(-x+1)^{M(M+1)/2}+\beta(-x+1)^{(M-2)(M-1)/2}. \tag{4}$$

The choice of 1 in $(\pm x+1)$ in (3) and (4) is arbitrary, but this choice causes no loss of generality.

From (1) and (2) we obtain

$$\theta(x)m_l(k,x,\alpha,N)=\sum_{n=0}^{N}\left(\frac{i}{k}\right)^n\left[\frac{(N+n)!(x+1)^{N(N+1)/2-n}}{2^n n!(N-n)!}+\alpha\theta(N-\tfrac{5}{2}-n)\frac{(N+n-2)!(x+1)^{(N-2)(N-1)/2-n}}{2^n n!(N-n-2)!}\right]\frac{1}{P(x,\alpha,N)}, \tag{5}$$

and

$$\theta(-x)m_r(k,x,\beta,M)=\sum_{n=0}^{M}\left(\frac{-i}{k}\right)^n\left[\frac{(M+n)!(-x+1)^{M(M+1)/2-n}}{2^n n!(M-n)!}+\beta\theta(M-\tfrac{5}{2}-n)\frac{(M+n-2)!(-x+1)^{(M-2)(M-1)/2-n}}{2^n n!(M-n-2)!}\right]\frac{1}{Q(x,\beta,M)}. \tag{6}$$

Using (5) and (6), we can write the physical solutions as

$$\psi_l(k,x,\alpha,\beta,c,M,N)=\theta(x)T(k)e^{ikx}m_l(k,x,\alpha,N)+\theta(-x)[e^{ikx}m_r(-k,x,\beta,M)+L(k)e^{-ikx}m_r(k,x,\beta,M)],$$

and

$$\psi_r(k,x,\alpha,\beta,c,M,N)=\theta(x)[e^{-ikx}m_l(-k,x,\alpha,N)+R(k)e^{ikx}m_l(k,x,\alpha,N)]+\theta(-x)T(k)e^{-ikx}m_r(k,x,\beta,M),$$

where the transmission and reflection coefficients are to be determined from the boundary conditions

$$\left(\lim_{x\to 0^+}-\lim_{x\to 0^-}\right)\begin{bmatrix}\psi_l(k,x,\alpha,\beta,c,M,N)\\ \psi_r(k,x,\alpha,\beta,c,M,N)\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix},$$

and

$$\left(\lim_{x\to 0^+}-\lim_{x\to 0^-}\right)\frac{d}{dx}\begin{bmatrix}\psi_l(k,x,\alpha,\beta,c,M,N)\\ \psi_r(k,x,\alpha,\beta,c,M,N)\end{bmatrix}=c\lim_{x\to 0}\begin{bmatrix}\psi_l(k,x,\alpha,\beta,c,M,N)\\ \psi_r(k,x,\alpha,\beta,c,M,N)\end{bmatrix}.$$

Hence we obtain

$$T(k)=2ik/D(k,\alpha,\beta,c,M,N),$$

and

$$L(k)=E(k,\alpha,\beta,c,M,N)/D(k,\alpha,\beta,c,M,N),$$

and

$$R(k)=E(-k,\alpha,\beta,c,M,N)/D(k,\alpha,\beta,C,M,N),$$

where we have defined

$$D(k,\alpha,\beta,c,M,N)\equiv(2ik-c)m_l(k,0,\alpha,N)m_r(k,0,\beta,M)+m_r(k,0,\beta,M)dm_l(k,0,\alpha,N)/dx - m_l(k,0,\alpha,N)dm_r(k,0,\beta,M)/dx, \quad (7)$$

and

$$E(k,\alpha,\beta,c,M,N)\equiv cm_l(k,0,\alpha,N)m_r(-k,0,\beta,M)+m_l(k,0,\alpha,N)dm_r(-k,0,\beta,M)/dx - m_r(-k,0,\beta,M)dm_l(k,0,\alpha,N)/dx. \quad (8)$$

If we let

$$c+[M-(M-1)\beta]/(1+\beta)+[N-(N-1)\alpha]/(1+\alpha)=0, \quad (9)$$

both $D(k,\alpha,\beta,c,M,N)$ and $E(k,\alpha,\beta,c,M,N)$ become independent of α and β; this can be seen by use of (5), (6), and (9) and by differentiation of (7) and (8) with respect to one of the parameters α and β. Thus, although the family of potentials $V(x,\alpha,\beta,c,M,N)$ still contains one of the parameters α and β as an arbitrary parameter, the corresponding scattering matrix becomes independent of both α and β.

The previously published nonuniqueness examples in the one-dimensional inverse quantum scattering are all special cases of the family $V(x,\alpha,\beta,c,N,M)$ considered here: $c=-2$, $M=1$, $N=1$ [6–8]; $c=-1$, $M=3$, $N=3$ [7,8]; $c=0$, $N=1$, $M=2$ [8,9]; $c=1$, $N=1$, $M=3$.[10]

As a special case,[11] let $M=1$; if we set $c+1/(1+\beta)+[N-(N-1)\alpha]/(1+\alpha)=0$, we obtain

$$D(k,\alpha,\beta,c,N;1)=2ik+\sum_{n=0}^{N-1}\left(\frac{i}{k}\right)^n\left[c-\frac{N(N-1)}{n+1}\right]\frac{(N+n-1)!}{2^n n!(N-n-1)!}$$

and

$$E(k,\alpha,\beta,C,N,1)=\sum_{n=0}^{N-1}\left(\frac{i}{k}\right)^n\frac{(c-n)(N+n-1)!}{2^n n!(N-n-1)!}.$$

The ambiguities in the one-dimensional inverse scattering are also studied by Sabatier with the use of the Darboux-Bäcklund transformation.[12–14] The nonuniqueness arises from the zero-energy poles, which are related to the value of the scattering matrix at zero energy.[14] For the families of potentials considered here, we have $T(k)=O(k^{N+M-1})$, $R(k)=\pm 1+O(k)$, and $L(k)=\pm 1+O(k)$ as $k\to 0$. The nonuniqueness arises from the double or higher-order zeros of the transmission coefficient at $k=0$ or the unit value of the reflection coefficients at $k=0$,[8,13–15] and specifying the ratio $m_l(k,x)/m_r(k,x)$ at $k=0$, $x=0$ uniquely specifies the parameter and hence removes the nonuniqueness.[8,13]

When the parameters α and β are nonnegative, the potentials considered here are positive everywhere and hence they do not support any bound states.

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