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Coherence, Chaos, and Broken Symmetry in Classical, Many-Body Dynamical Systems

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It is argued in the context of coupled quadratic maps that macroscopically chaotic states do not occur in many-body systems with local interactions and random initial conditions. Such systems can exhibit chaos, but only locally; their collective behavior is periodic or stationary. The phase diagram for the coupled-map system as a function of control parameter and noise is presented, and the universality classes of the phase transitions identified.

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Many model dynamical systems (either maps or small sets of coupled ordinary differential equations) originally derived as approximations to many-body systems such as partial differential equations (PDE's) or cellular automata (CA), have been shown to exhibit nonstationary (periodic or chaotic) behavior. It is commonly believed that the full many-body system has spatially extended modes whose time dependence is well described by the nonstationary states of these "zero-dimensional (0D) approximants" (i.e., approximants consisting of a finite number of modes). In certain probabilistic CA, e.g., it has been argued[1] that the spatial average (i.e., the $k=0$ Fourier mode) of the variables in the system can vary periodically or chaotically in time, and exhibit all other behavior familiar from single-variable (0D) maps.[2] Numerical evidence for similar phenomena in PDE's has been reported.[3]

In this paper we study the effect on the nonstationary states of dynamical, many-body systems of the fluctuations neglected in 0D approximations. Our results are derived for noisy[4] coupled maps[5] on regular d-dimensional lattices, though they should apply equally well to other dynamical systems. Specifically, we consider the model

$$\psi_{n+1}(i)=f\left[\frac{1}{2d+1}\sum_{i'}\psi_n(i')\right]+\eta_n(i). \tag{1}$$

Here $\psi_n(i)$ is the variable at the nth time step on the ith site of a d-dimensional hypercubic lattice with N sites; the function f defines the map, which is assumed to undergo, as a control parameter r is increased, a bifurcation sequence leading to chaos[2] at a critical value, r_c; $\eta_n(i)$ represents random noise with local Gaussian correlations of width σ: $\langle\eta_n(i)\eta_{n'}(j)\rangle=\sigma^2\delta_{n,n'}\delta_{i,j}$; i' is summed over i and its 2D near neighbors. Our main results are illustrated in the schematic phase diagram (Fig. 1) appropriate to the quadratic map,[6] $f(x)=rx(1-x)$, with $d>1$. We now briefly summarize its central features.

First recall that with increasing r the single variable (0D) quadratic map with $\sigma=0$ undergoes[2] an infinite sequence of period-doubling bifurcations leading, at

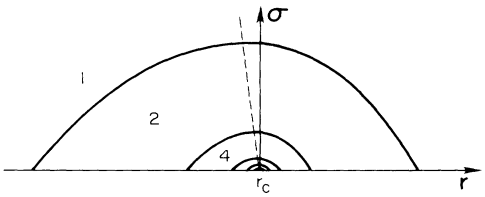

FIG. 1. Schematic phase diagram for coupled quadratic maps for $d>1$. See text for discussion.

$r=r_c=3.569\ldots$, to chaos. As r increases beyond r_c, an inverse bifurcation sequence of bands (interrupted by periodic windows) occurs.[2,7] For $\sigma=0$ and $r<r_c$ the coupled map system has[5] linearly stable, spatially uniform 2^m-cycles which correspond precisely to those of 0D, and exhibit the same period-doubling cascade. Numerical simulations show clear evidence that these cycles persist for sufficiently small noise. For $\sigma>0$ and $N\to\infty$ they are, however, spatially uniform only in that the noise-averaged value $\langle\psi_n(i)\rangle$ is independent of i, and periodic in that $\langle\psi_n(i)\rangle$ (or, equivalently, the spatial average $\sum_i\psi_n(i)/N$ for specific noise variables) executes a 2^m-cycle in n. These states are connected through period-doubling transitions, argued below to belong, for $\sigma>0$, in the universality class of the d-dimensional kinetic Ising model; for $\sigma=0$ they are mean-field-like. The successive period-doubling transitions break the discrete time-translation symmetry of (1) progressively; i.e., the 2^m-cycle is invariant only under time translations of 2^m time steps. For any fixed σ, however, the period-doubling sequence terminates at some maximum period which increases with decreasing σ, diverging as $\sigma\to 0$; thus states of arbitrarily long period occur for small enough noise. (This is closely analogous to the "bifurcation gap" predicted[7] for noisy 0D maps.)

As r increases beyond the point where this maximum occurs, the system undergoes an inverse sequence of period-doubling transitions, becoming stationary (a 1-cycle) as r approaches 4. There is, strikingly, no distinct chaotic phase, i.e., no phase, even for $\sigma=0$, in which the time-translation symmetry is completely broken, so that $\langle\psi_n(i)\rangle$ varies chaotically in time. The system has a "chaotic" regime, but the chaos is purely local, not collective. That is, the evolution of any *individual* variable is chaotic, as measured by the positivity of a Lyapunov exponent, λ, computed, e.g., from the sensitivity[8] to initial conditions of $\psi_n(i)$ for a particular i; however, the collective behavior of the system remains periodic. A schematic $\lambda=0$ line is shown (dashed) in Fig. 1. To its right (left), $\lambda>(<)\,0$, and the system is locally chaotic (periodic or stationary). Unlike period-doubling boundaries, this line does *not* represent a broken symmetry or phase transition. The absence of a collective effect means that λ should go through zero in a way governed by an effective 0D map. For $\sigma>0$, λ must change sign *analytically* as a function of r, given that no 0D map can have singularities for $\sigma>0$. (This would be tantamount to[4b,9] a 1D Ising model having a phase transition at finite temperature.)

For $\sigma=0$ the system if often multistable[5] (i.e., different initial conditions produce distinct stable states, only the most robust of which is shown in Fig. 1). As argued below, however, such multistability cannot persist at generic points of Fig. 1 for $\sigma>0$.

To understand these results, first note that if the system is spatially uniform and $\sigma=0$, (1) reduces, for an d, to the single-variable map. Thus for $r<r_c$ the coupled map system has spatially uniform 2^m-cycle states which correspond precisely[5] to those of 0D. It is easy to show[5] that for any d and sufficiently small σ, these uniform 2^m-cycles are linearly stable with respect to spatial fluctuations. To establish the global stability of the cycles, however, one must consider the *nonlinear* terms of (1). It is well known[4b,9] that models such as (1) in d dimensions are identical to $(d+1)$-dimensional equilibrium Ising-type models, and that, of the nonlinear fluctuations in Ising problems, the potentially most disruptive are "domain-wall" excitations. To see the effect of such excitations here, consider the case $m=1$ for $\sigma=0$, and denote the fixed point values by $\psi_1^\times$ and $\psi_2^\times$. A finite domain, linear size R, of $\psi_{1(2)}^\times$ immersed in a sea of $\psi_{2(1)}^\times$ will, given the stability of both $\psi_1^\times$ and $\psi_2^\times$ under the once iterated map f^2, shrink with time for all $d>1$, because of its finite curvature; it will disappear, as is familiar from equilibrium Ising models,[10] in a time proportional to R^2. As for Ising models, sufficiently small noise alters only the proportionality constant[10] (and $\psi_1^\times$ and $\psi_2^\times$), not the qualitative phenomenology; hence for $d>1$, 2-cycles are stable with respect to domain formation for sufficiently small σ. When σ becomes roughly comparable to $|\psi_1^\times-\psi_2^\times|$, the distinction between $\psi_1^\times$ and $\psi_2^\times$ blurs, and the 2-cycle disorders, i.e., makes a transition to the 1-cycle.

For $m>1$, the stability of 2^m-cycles for sufficiently small σ and $d>1$ has been checked numerically. Again, when σ becomes roughly comparable to the smallest spacing between cycle values, the 2^m-cycle undergoes a transition to a $2^{(m-1)}$-cycle. Thus for any given σ there is a maximum period, $2^m_{\max}$, for $r<r_c$; $m_{\max}$ increases with decreasing σ, so that (Fig. 1) states of arbitrarily high period occur near the point $r=r_c$, $\sigma=0$.

As in equilibrium Ising systems, domain walls are more important for $d\le 1$. Since in $d=1$ walls have no curvature, droplets need not shrink with time,[5] but proliferate, for $\sigma>0$, destabilizing any cycle with $m\ge 1$: For $d\le 1$ only stationary (1-cycle) states are possible for $\sigma>0$,[11] though *metastable* states with $m\ge 1$ occur.[5]

In the chaotic regime of the 0D map, $r>r_c$, it is straightforwardly shown that the spatially uniform chaotic state is linearly unstable. This result reflects the enormous sensitivity to initial conditions of the chaotic regime, which is characterized by a positive Lyapunov exponent, λ, and in which spatial nonuniformities resulting from either nonuniform initial conditions or noise grow exponentially. This implies the destruction of spatial coherence at long times, except in the noiseless case for very special (e.g., spatially uniform) initial conditions.

We now argue heuristically that this sensitivity precludes not only the spatially uniform chaotic state, but *any* state whose spatial average varies chaotically in time. Since any spatial average over uncorrelated regions will necessarily produce a time-independent result, it suffices to show that spatial correlations in any locally chaotic state can extend only over some finite distance, say ξ. To show this, set $\sigma=0$. Now imagine that at some time, say $n=0$, $\psi_0(i)$ is perturbed by an infinitesimal amount, $\delta\psi_0(i)$; the evolution then proceeds uninterruptedly according to (1). Since by assumption each variable evolves chaotically in time, the deviation, $\delta\psi_n(i)$, of $\psi_n(i)$ from the value it would have assumed in the absence of the perturbation, grows like $\delta\psi_n(i) \sim \delta\psi_0(i)e^{\lambda n}$, where $\lambda>0$ is the appropriate Lyapunov exponent. When $\delta\psi_n(i)\sim 1$, i.e., when $n \ge n^\times \equiv -\ln[\delta\psi_0(i)]/\lambda$, $\psi_n(i)$ has lost all memory of the value it would have achieved had $\delta\psi_0(i)$ been zero. Since, for near-neighbor coupling, information cannot be transmitted from site to site with speed greater than unity, any other site, j, evolves independently of $\delta\psi_0(i)$ until time $n=|i-j|$, when it first learns of the perturbation. If $|i-j|>n^\times$, then by the time j receives this information, i is completely decorrelated from the value it would have had in the absence of the perturbation. Thus i and j have been decorrelated by the perturbation. [To be more precise, recall that for $r>r_c$ the 0D map exhibits an inverse bifurcation sequence,[2,7] at each level of which the chaotic attractor consists of 2^m ($m=\infty,\ldots,2,1,0$) bands of values of ψ, separated by gaps. The variable moves chaotically within each band, but *periodically* from band to band. Thus for $\delta\psi_0(i)\ll 1$ the variable at i always remains in the same band as the one at j, but the two variables are otherwise completely decorrelated. It is this decorrelation *within a given band* that we refer to when we discuss the decorrelating effect of perturbations.]

If $\delta\psi_0(i)$ is due to noise, σ, then similar perturbations occur at every instant, constantly preventing any correlation between i and j. Hence the maximum distance over which the $\psi_n(i)$ can maintain spatial coherence is $n^\times$; with $\delta\psi_0(i)\sim\sigma$ one obtains, for $\sigma\ll 1$, the rough bound: $\xi \le -\ln\sigma/\lambda$. Even for $\sigma=0$, however, random initial conditions are a source of decorrelating perturbations which enforce $\xi \le -\ln\delta\psi_0(i)/\lambda$, where $\delta\psi_0(i)$ is a measure of the randomness. Thus long-range coherence is prohibited except for $\sigma=0$ and special (e.g., spatially periodic[4]) initial conditions. A spatial average over the various uncorrelated volumes of size ξ therefore produces, for each band, a fixed, n-independent "average" value of $\sum_i \psi_n(i)/N$ (or $\langle\psi_n(i)\rangle$). This suggests that for $d>1$ and σ smaller than the gaps, the system should execute periodic motion among these 2^m average values, i.e., is in a 2^m-cycle state! Hence the spatial average does not become, even for $d>1$, chaotic for $r>r_c$, but remains periodic, undergoing an inverse bifurcation sequence which produces a stationary state at large r, as in Fig. 1. (Note that the inverse bifurcations occur, because of the intersite coupling, at values of r different from those of the 0D map, and that, for $d\le 1$, domain walls ensure that, as for $r<r_c$, only 1-cycles occur for $\sigma>0$.)

We have verified this picture by extensive numerical simulation in 1D and 2D, for both $r>r_c$ and $r<r_c$. Figure 2, e.g., shows a histogram of values of the spatial average, $\sum_i\psi_n(i)/N$, for $d=2$, $\sigma=0.01$, and $r=3.572$, where the 0D ($N=1$) map has eight chaotic bands [Fig. 2(a)]. With increasing N the histogram sharpens, yielding eight distinct spikes [Fig. 2(b)] whose widths decrease consistent with the expected $N^{-1/2}$ behavior.

None of the periodic "windows" of stable states which occur[2] in the chaotic regime of the 0D map appear in Fig. 1. While at $\sigma=0$ we did find such spatially uniform cycles in simulations which started from nearly uniform initial states,[5] we also found, consistent with earlier work,[5] that the many-body system is often multistable. Starting from more random initial conditions, e.g., we always found one of the (spatially nonuniform) 2^m-cycles just discussed. The 2^m-cycles are, moreover, more stable than the window cycles; i.e., a flat domain wall separating these two states moves preferentially to eat up the window cycle. Thus, according to our numerics, for any $\sigma>0$ the window cycles become *metastable* and only the 2^m-cycles are stable. This absence of multistability for $\sigma>0$ is general: One expects[12] that for local, spatially symmetric interactions, two states can be equally stable only by accident or symmetry. Since any Gaussian noise eventually moves the system into the most stable state,

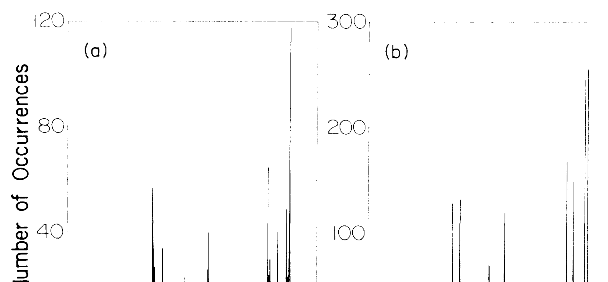

FIG. 2. Histogram showing (a) eight chaotic bands for 0D quadratic map at $r=3.572$, $\sigma=0$; (b) eight spikes for 100×100 lattice at the same r and $\sigma=0.01$.

multistability is prohibited at generic points of parameter space.

To understand the nature of the various phase transitions in Fig. 1, consider the transition between the 1- and 2-cycles. By iterating (1) once, one can easily show that this equation is simply a discretized, simultaneously updated version of the ordinary time-dependent Ginzburg-Landau equation[13] for describing the liquid-gas transition near its critical point. We conclude, therefore,[14] that the period-doubling transitions belong (so long as they remain continuous) in the universality class of the kinetic Ising model.

Since the results obtained here are based on rather general arguments, we hypothesize that they also apply to CA and PDE's. Thus, in neither of these systems should collective chaos, i.e., chaos in the amplitude of an extended (e.g., Fourier) mode occur, except for zero noise and special initial conditions. Hence, e.g., predictions[1] that probabilistic CA (PCA) ought to have states in which the spatial average of the variables varies chaotically in time are artifacts[15] of the neglect of spatial fluctuations. We emphasize that the intrinsic instability of collectively chaotic states should hold equally well for systems which achieve chaos through avenues other than period doubling.

It should be clear from the foregoing that the absence of collective chaos will only be manifest experimentally in measurements on length scales long compared to the correlation length, ξ. For example, in Rayleigh-Benard experiments, ξ is of the order of the roll size.[16] Chaos in a macroscopic variable has been observed[17] in cells with only a few rolls—too small for the averaging effect we predict. Experiments on large-aspect-ratio cells containing many rolls would provide a test of our ideas.

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[6]For $\sigma > 0$ one must modify $f(x)$ at large x to keep the fields $\psi_n(i)$ roughly confined to the unit interval. Any f with $0 < f(x) < 1$ for all $|x| > 1$ accomplishes this.

[7]J. P. Crutchfield and B. A. Huberman, Phys. Lett. **77A**, 407 (1980).

[8]For $\sigma > 0$, λ can presumably be defined in terms of the noise-averaged quantity $\langle |\delta\psi_n(i)|^2\rangle$ where $\delta\psi_n(i)$ is the change in $\psi_n(i)$ resulting from a tiny perturbation, $\delta\psi_n(i)$, at $n=0$. It cannot be computed from $\langle\delta\psi_n(i)\rangle$, as was suggested in Ref. 4b for 0D, since this will always produce a $\lambda < 0$. We thank B. Shraiman and P. C. Martin for helpful discussions on this point.

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[11]This result for $d \leq 1$ contradicts a statement in Ref. 4b, where it is argued that, even for the noisy 0D map, $\langle\psi_n\rangle$ need not approach a constant as $n \to \infty$. Similar statements in Ref. 7 are, however, consistent with our picture since the noise there is bounded, not Gaussian. Bounded noise need not destroy long-range order, even in statistical mechanics, where 0D systems with bounded noise can undergo phase transitions, an impossibility with the more physical unbounded noise.

[12]Certain spatially asymmetric interactions permit multistability at generic points in parameter space. See, e.g., A. L. Toom, in *Multicomponent Random Systems,* edited by R. L. Dobrushin, Advances in Probability Vol. 6 (Dekker, New York, 1980); P. Gacs, to be published; C. H. Bennett and G. Grinstein, Phys. Rev. Lett. **55**, 657 (1985).

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[15]This explains why in simulations of PCA (Ref. 14) we saw, instead of the full bifurcation sequence leading to chaos, predicted in Ref. 1, bifurcation sequences that achieved a maximum period. We believe that indications of chaotic behavior in those simulations were artifacts of finite size.

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