Quantum Conductance in Networks

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We consider the quantum transport in networks. Arguments similar to those for the quantum Hall effect show that the averaged transport coefficients are quantized. Numerical calculations for a three-loop network yield the values 0, 1, and -1, depending on the fluxes threading the loops and the quantum state of the net. We characterize the conductance properties of such networks. We also discuss general properties of the transport coefficients in general multiloop networks.

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It is known that there is a range of circumstances where the Hall conductance, at low temperatures, is a nonzero integer.[1] It is natural to inquire whether there are other systems with integer nonzero conductances. As we shall explain, networks are such systems: A network with L loops has $L(L-1)/2$ integer conductances which characterize the quantum state of the system and reflect its multiconnectivity. Like the Hall conductance, they are nondissipative and can be either holelike or electronlike, but unlike the Hall effect this does not reflect any band-structure properties.

The transport coefficients of the network, $2\pi g_{lm}$, are defined as the charge transported around loop l when the flux threading the mth loop, ϕ_m, increases adiabatically by 2π, the unit of quantum flux. Within linear-response theory, it turns out[2] that this is equivalent to the (time-averaged) ratio of the current in loop l to an infinitesimal emf acting on loop m.[3,4] We shall concentrate on the cases where the network has three loops and l and m are distinct. We shall also assume throughout that the fluxes are changed sufficiently slowly for the adiabatic limit to hold. In particular the energy levels of the network are assumed to have nonvanishing gaps and we exclude situations where levels cross. Under these conditions, which guarantee no dissipation (dissipation arises when $l=m$ and the adiabatic limit does not hold), the nondiagonal conductances have nonlocal features. Also, the quantum (coherence) effects discussed below require temperatures which are low compared with a typical gap energy. Since energy gaps scale like (length)$^{-2}$ this favors small networks. This dictates temperatures in the millikelvin range and emf in microvolts for mesoscopic networks. Quantum coherence effects associated with the dissipative conductance in single mesoscopic loops, including nonlocal effects, are discussed by Sharvin and Sharvin.[5] As yet, there are no experiments nor theory on the transport coefficients in two- or three-loop networks.

Consider, for example, a three-loop network made of mesoscopic, thin (metallic) wires (Fig. 1). Each loop is threaded by an independent flux tube ϕ_j, $j=1,2,3$. In comparison with the Hall effect, ϕ_3 plays the role of the magnetic field on the sample, ϕ_1 can be thought of as a time-dependent flux replacing a battery, and ϕ_2 is the analog of Laughlin's flux tube. g_{12} is then the analog of the Hall conductance. Because of the analogy one may expect that g_{12} will be quantized and will be a nontrivial (antisymmetric) function of ϕ_3. This, as we shall see, is essentially correct provided that suitable averaging is introduced: Let

$$\langle g_{lm}\rangle(\phi)\equiv\frac{1}{2\pi}\int_0^{2\pi}d\phi_m\, g_{lm}(\phi)$$

be the conductance averaged over the flux in the current loop m (ϕ denotes collectively the three fluxes). $\langle g_{12}\rangle(\phi_3)$ (or any other permutation of 1,2, and 3) is an antisymmetric steplike function of ϕ_3 with steps at integer heights. This holds in great generality (i.e., even with electron-electron interaction, and also for thick wires and for more complicated networks) provided the system is in a pure quantum state which does not become degenerate as ϕ_1 and ϕ_2 are varied. It is a consequence of the fact that Kubo's formula for the (averaged) conductance has a topological interpretation being a first Chern number.[2,6-10]

Here we shall describe parts of our numerical results and sketch the general theoretical structure. Details shall be presented elsewhere.[11]

From the theory of superconducting networks[12] it is known that the analysis of the Schrödinger equation for the network of Fig. 1 (with one-dimensional wires) reduces to the study of 5×5 matrices (5 is the number of

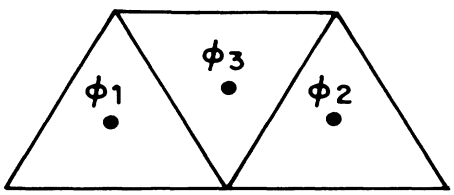

FIG. 1. Three-loop network with seven edges and five vertices. Each loop is threaded by a flux tube. The Hamiltonian for the network with point junctions is a 5×5 matrix. This network has nonzero quantized conductances.

vertices in the network) of the tight-binding type. It is therefore not surprising that the computation of the transport coefficients of the network also reduces to a (5×5)-matrix problem. The details of the reduction shall be given elsewhere.[11] In the matrix description the wave function sits on the vertices of the network. Consider the tight-binding Hamiltonian

$$H(v,v':\phi)=n_v\delta_{vv'}+(1-\delta_{vv'})\sum_b[v,b][v',b]\exp(-i[v,b]\gamma_b). \tag{1}$$

v and v' are vertex indices. n_v is the "coordination number" of the vertex v. b is a (directed) edge index, and $[v,b]$ is the incidence matrix, i.e., $[v,b]$ is 1 if the edge b points into v, -1 if it points out of v, and 0 otherwise. $\gamma_b\equiv\int_b A$, where A is the vector potential associated to the fluxes ϕ. The lengths of all the edges b is set equal to one. $H(\phi)$ is identical to the de Gennes–Alexander[12] network Hamiltonian except on the diagonal. This slight modification makes it somewhat easier to handle. (The de Gennes–Alexander Hamiltonian gives an implicit eigenvalue problem.) Because of our interest in topological invariants the difference is presumably immaterial.

Diagonalizing the Hamiltonian one finds that the Chern number in the ground state, defined by Eq. (3) below, is

$$\langle g_{12}\rangle(\phi_3)=\begin{cases}0 \text{ for } -\pi/3<\phi_3 \bmod 2\pi<\pi/3\ ,\\ 1 \text{ for } \pi/3<\phi_3\bmod 2\pi<\pi,\\ -1 \text{ for } -\pi<\phi_3\bmod 2\pi<-\pi/3.\end{cases} \tag{2}$$

For the excited states one finds qualiatively similar, i.e., nontrivial, antisymmetric, periodic steplike functions that take the values 0, 1, and -1. One also finds that $\langle g_{13}\rangle(\phi_2)=\langle g_{23}\rangle(\phi_1)=0$ identically for all the states. Because of the topological nature of the results the fact that the network is made of three equilateral triangles is immaterial and one finds the same qualitative features in any network which is a deformation of Fig. 1.

To get a complete description and insight into the results we have to introduce some formalism. This is also necessary in order to describe the actual computation.

$H(\phi)$, the exact Schrödinger operator of the network, depends parametrically on the fluxes ϕ. For fixed ϕ, it has discrete spectrum. Because of the periodicity in the fluxes the parameter space can be identified with T^3, the three-torus, i.e., we can identify ϕ_j with $\phi_j+2\pi$.[13]

Let $P(\phi)$ denote a projection on a spectral subspace of $H(\phi)$ and C be a closed, two-dimensional surface in T^3 (equal to a closed two-chain). Suppose that $P(\phi)$ is smooth on C. It is a standard fact that the Chern number,[2,6] $\mathrm{Ch}(P,C)\equiv(i/2\pi)\int_C\mathrm{Tr}[dP\,P\,dP]$, is an integer.

If the initial state of the system is given by $P(\phi)$ and there is no level crossing on $T_{lm}(\phi)$ (the two-dimensional slice of the three-torus going through ϕ and indexed by l and m), then Kubo's formula reads[2,7–10] (see Ref. 10 for a rigorous derivation)

$$\langle g_{lm}\rangle(\phi)=\mathrm{Ch}(P,T_{lm}(\phi)). \tag{3}$$

It follows[11] that $\langle g\rangle_{lm}(\phi)$ is gauge invariant, periodic in ϕ, antisymmetric in l and m, and independent of ϕ_l and ϕ_m, and is quantized to be an integer. Also, in the absence of magnetic fields besides ϕ, which we shall assume, time reversal leads to the Onsager relation $\langle g_{lm}\rangle(\phi)=-\langle g_{lm}\rangle(-\phi)$.

It is known that the Chern numbers are closely related to degeneracies. Let D_q be the set of points where the qth gap in the energy spectrum closes. According to the von Neumann–Wigner theorem[14] D_q is a discrete set. The second homology group of T^3/D_q is spanned by three two-tori, T_{12}, T_{23}, and T_{31}, and $|D_q|$ oriented two-spheres $S(\delta)$ that surround $\delta\in D_q$. An arbitrary closed two-chain in T^3/D_q can be written as a sum of the basic spheres and tori with integer coefficients. This relation lifts to a relation for the Chern numbers. It follows that the set of $3+\sum|D_q|$ Chern numbers contains all the information about the net.

Relations among the basic Chern numbers follow from the following facts:

(1) $\sum_{\delta\in D_q}S(\delta)$ is homologous to zero, so that $\sum_{\delta\in D_q}\mathrm{Ch}(P_q,S(\delta))=0$, for all q, where P_q is the projection on the spectral subspace with energies up to the qth level.

(2) The set D_q is invariant under inversion $\phi\Longrightarrow-\phi$, and for any δ in D_q, $\mathrm{Ch}(P_q,S(\delta))=\mathrm{Ch}(P_q,S(-\delta))$.

For any closed two-chain c which is invariant under inversion, $\mathrm{Ch}(P_q,c)=0$. (4) If $H(\phi)$ is a periodic $n\times n$ matrix, P_n is the identity and so all its Chern numbers vanish.

For the Hamiltonian of Eq. (1) the set of points of degeneracy and their Chern numbers are given in Table I. Because of (2) above only points in the half-cube with $0\le\phi_3\le\pi$ are listed. For the three basic tori we find

$$\begin{aligned}&\mathrm{Ch}(P_q,T_{23}(\phi_1))=\mathrm{Ch}(P_q,T_{13}(\phi_2))=0,\\ &\mathrm{Ch}(P_q,T_{12}(\pi/2))=\delta(q,1).\end{aligned} \tag{4}$$

This gives a complete characterization of the nondissipa-

TABLE I. Chern numbers for spheres surrounding points of degeneracy in the network of Fig. 1. $\alpha\equiv\arccos(\frac{1}{2}-\sqrt{2})$.

Gap	Coordinates	Chern number
1	$(\frac{2}{3},\frac{2}{3},\frac{1}{3})\pi$	1
1	$(\frac{4}{3},\frac{2}{3},1)\pi$	-1
2	$(-\alpha,\alpha,0)$	1
2	$(\alpha,\alpha,\alpha/2)$	-1
3	$(\frac{4}{3},\frac{4}{3},\frac{2}{3})\pi$	1
3	$(\frac{2}{3},\frac{4}{3},1)\pi$	-1

tive averaged conductance of the three-loop network of Eq. (1).

Equation (3), which relates the Chern numbers with the transport coefficients, is known to hold for the full Schrödinger equation of the network. We shall now describe how to extend this to matrix Hamiltonians. We shall consider here the case of matrix Hamiltonians where the wave function sits on the bonds of the network. The case where it sits on vertices is more complicated and shall be dealt with elsewhere.[11]

In a one-dimensional Schrödinger equation of a network the wave function is $\Psi_b(x_b)$, $0 \le x_b \le 1$. The coordinate x_b is measured in the b direction. On b, Ψ solves the one-dimensional Schrödinger equation in a gauge potential and therefore has the form

$$\Psi_b(x) = [T(\psi_+, \psi_-)](x) \equiv \exp[i\gamma_b(x)][\psi_+(b)\exp(ikx) + \psi_-(b)\exp(-ikx)], \tag{5}$$

where $\gamma_b(x) \equiv \int^x A$. $\psi_+(b)$ and $\psi_-(b)$ denote the amplitudes of the forward- and backward-moving waves on the bond b. The linear operator T is introduced for later purposes. A vertex with n_v edges connected to it is described by a unitary $n_v \times n_v$ scattering matrix, S_v,[15] which maps the incoming waves on the outgoing waves, $S_v \Psi_{v,\text{in}} = \psi_{v,\text{out}}$. $\Psi_{v,\text{in/out}}$ are n_v-dimensional vectors of complex numbers:

$$\begin{aligned} \Psi_{v,\text{in}}(b) &\equiv \delta(1,[v,b])\psi_+(b)\exp[ik + i\gamma_b] + \delta(-1,[v,b])\psi_-(b), \\ \Psi_{v,\text{out}}(b) &\equiv \delta(1,[v,b])\psi_-(b)\exp[-ik + i\gamma_b] + \delta(-1,[v,b])\psi_+(b). \end{aligned} \tag{6}$$

b runs over the n_v edges associated to v. The unitarity of S_v guarantees that current is conserved at each vertex.

A basic tool is this: Consider $T: H_1 \Rightarrow H_2$, where $H_{1,2}$ are two Hilbert spaces (not necessarily of the same dimension). Let Q be an orthogonal projection on H_2, and suppose that $TT^*Q = Q$ (T^* denotes the adjoint of T). Suppose that $Q\,dT\,T^*$ is smooth and globally defined. Then $P \equiv T^*QT$ is a projection on H_1 and the Chern numbers for the two projections coincide.

We apply this to H_1, the finite-dimensional complex vector space, and H_2, the Hilbert space of functions. The map T from C^2 to $L^2[0,1]$ is given by Eq. (5). The scalar product in C^2, induced by the scalar product in L^2, has the "Riemann metric" A where

$$A_{ij} \equiv \delta_{ij} + [(1-\delta_{ij})/k]\exp(-ik\epsilon_{ij})\sin(k), \quad i,j = 1,2.$$

ϵ_{ij} is the completely antisymmetric tensor. The metric is independent of the gauge field and is nonsingular provided $k \neq 0$. One finds[11] for dT^*T

$$dT^*T = d(A^{-1})^* A - i(A^{-1})^* \begin{pmatrix} \int [d\gamma(x) + x\,dk]dx & \int [d\gamma(x) - x\,dk]\exp(-2ikx)dx \\ \int [d\gamma(x) + x\,dk]\exp(2ikx)dx & \int [d\gamma(x) - x\,dk]dx \end{pmatrix}. \tag{7}$$

γ is linear in ϕ, so that $d\gamma$ is independent of ϕ. Q, k, A^{-1}, and $Q\,dT\,T^*$ are all smooth in ϕ provided no levels cross and k does not vanish. This establishes the equality of the Chern numbers.

In summary, networks have quantized averaged conductances which are nontrivial in networks with three or more loops. The computation of Chern numbers for network Hamiltonians with one-dimensional connecting links reduces to the study of finite matrices. Finally, homology provides a convenient and compact way of presenting the conductance functions of networks.

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