Density and Deceleration Limits in Tapered Free-Electron Lasers

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The competition between the bunching force of the ponderomotive potential and the repulsive force of the self electrostatic potential in a tapered free-electron laser limits the beam density, the deceleration rate due to tapering, and the local electric-field spatial exponentiation rate. The limit on the spatial exponentiation rate is below that predicted by linear theory at the transition from the Compton to Raman regimes. Furthermore, this limit is independent of the form of the particle energy distribution function.

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The intrinsic efficiency of free-electron lasers with constant wiggler parameters is limited to a few percent.[1] The limitation results from the fact that the maximum change in particle velocity is of the order of the trapping velocity due to the ponderomotive potential which is much smaller than the initial beam velocity. It has been suggested[2,3] that the efficiency can be enhanced by adiabatic change of the wiggler parameters so as to gradually slow down the ponderomotive wave and any particles trapped in it. With proper tapering the efficiency can be improved dramatically.[2]

A fundamental limitation on this process is that the ponderomotive well cannot be decelerated so rapidly that particles spill over its sides, becoming untrapped.[2,3] This, in turn, imposes a minimum length on the interaction region if a given efficiency is desired. In this Letter it will be shown how this limit on deceleration rate depends on particle density when the repulsive self-electric-field force between particles is taken into account. In particular, for a given deceleration rate there is a maximum density of particles that can be decelerated. Likewise, for a given density there is a maximum deceleration rate that can be achieved. Furthermore, the combination of these limits sets an upper bound to the spatial exponentiation rate of the high-frequency fields.

To determine these limits the equation describing the axial motion of particles due to the combination of the ponderomotive and electric forces is considered,

$$m\gamma_0^3\ddot{z} = -q\frac{\partial}{\partial z}\phi(z) + \frac{\partial}{\partial z}\varepsilon_0\cos(kz) - m\gamma_0^3\dot{v}_0, \qquad (1)$$

where $z+\int^t v_0\,dt$ is the axial position of a particle of rest mass m and charge q, v_0 is the velocity of the ponderomotive well, $\gamma_0=(1-v_0^2/c^2)^{-1/2}$ is the relativistic factor (for simplicity we assume that the contributions to γ_0 from perpendicular motion are negligible), and ϕ is a potential which generates the axial electric field.

The quantity ε_0 is the amplitude of the ponderomotive potential,

$$\varepsilon_0 = -q^2E_rB_w/2\gamma_0mk_wk_rc^2,$$

where E_r and B_w are the radiation electric field and wiggler magnetic field, respectively, and $k=k_w+k_r$, where k_w and k_r are the wave numbers for the wiggler and the radiation fields, respectively. The last term on the right in Eq. (1) represents the inertial force due to deceleration with $\dot{v}_0=dv_0/dt<0$. In general, all the quantities in the definition of ε_0 depend on axial distance down the interaction length. It is assumed that this dependence is weak on the length scale of the beat wave and that these quantities can be considered as constants in what follows.

The potential ϕ, which generates the axial electric field, is a combination of the electrostatic potential and the axial vector potential. Both electric and magnetic perturbations are present because the charge perturbations are moving with speed v_0. The potential ϕ satisfies the modified Poisson equation,

$$\gamma_0^2\nabla_\perp^2\phi + \partial^2\phi/\partial z^2 = -4\pi qn.$$

In general, both the axial and perpendicular derivatives in the preceding are important. The case of relatively thick beams satisfying $\gamma_0^2/r_b^2 \ll k^2$ will be considered here. The opposite limit of a thin beam can be treated as well and leads to similar conclusions. For a thick beam the potential can be separated into two parts, $\phi=\bar{\phi}+\tilde{\phi}$, where $\bar{\phi}$ is independent of axial position z,

$$\gamma_0^2\nabla_\perp^2\bar{\phi} = -4\pi q\bar{n},$$

where $\bar{n}$ is the average density, and $\tilde{\phi}$ is periodic in z with period equal to that of the ponderomotive wave,

$$\partial^2\tilde{\phi}/\partial z^2 = 4\pi q(\bar{n}-n). \qquad (2)$$

Equilibrium solutions to Eqs. (1) and (2) are now sought. Particles satisfying Eq. (1) have a constant of motion

$$\varepsilon = (m\gamma_0^3/2)\dot{z}^2 + \varepsilon_0\psi(\xi),$$

where $\xi=kz$ is a normalized distance, and $\psi(\xi)$ is a dimensionless effective potential well,

$$\psi(\xi) = q\tilde{\phi}(\xi)/\varepsilon_0 - \cos\xi - \alpha\xi, \qquad (3)$$

and

$$\alpha = m\gamma_0^3 |\dot{v}_0| / k\varepsilon_0$$

measures the deceleration rate. The contribution of particles with energy ε to the local density follows from

$$n = \int \frac{2d\varepsilon f(\varepsilon)}{m\gamma_0^3 |\dot{z}|} = n(\psi), \tag{4}$$

where f is the distribution function and the integral is over those values of ε corresponding to particles trapped in the potential well $\varepsilon_0 \psi$.

If we adopt the previous normalizations for length and potential, Eq. (2) yields a second-order nonlinear differential equation for $\psi(\xi)$,

$$d^2\psi/d\xi^2 = \cos\xi + \bar{\eta} - \eta(\psi), \tag{5}$$

where

$$\bar{\eta} = 4\pi q^2 \bar{n} / k^2 \varepsilon_0 \tag{6a}$$

measures the average density, and

$$\eta(\psi) = 4\pi q^2 n(\psi) / k^2 \varepsilon_0 \tag{6b}$$

measures the local density. The boundary conditions that are applied to $\psi(\xi)$ are that $\psi(\xi) + \alpha\xi$ is periodic in ξ with period 2π.

A solution of Eq. (5) requires specification of the distribution function $f(\varepsilon)$. As an example, consider the simple case where f is constant for those values of ε corresponding to trapped particles and zero otherwise. In this case

$$\eta(\psi) = \eta_* \times \begin{cases} \sqrt{-\psi} \text{ for } \psi < 0 \\ 0 \text{ for } \psi \geq 0 \end{cases},$$

where η_* is a constant proportional to the constant distribution function.

A sample numerical solution to Eq. (5) for this case is shown in Fig. 1, where the effective potential and the local density are shown as functions of ξ. Particles are trapped in regions of ξ where $\psi < 0$. (Because the equations are unchanged by addition of a constant to $\tilde{\phi}$, one is free to pick $\psi = 0$ at the local maximum on the right.)

The solution was obtained in the following way. The value of η_* and the location of the local maximum on the right, ξ_0 [where $\psi(\xi_0) = \psi'(\xi_0) = 0$], were specified. Equation (5) was then integrated backwards in ξ for different values of $\bar{\eta}$ until one was found for which $\psi'(\xi_0 - 2\pi) = 0$. The corresponding value of α, the deceleration, was then determined from

$$\alpha = [\psi(\xi_0 - 2\pi) - \psi(\xi_0)]/2\pi. \tag{7}$$

This procedure was repeated for different values of ξ_0 and η_*, and the resulting values of α and $\bar{\eta}$ were plotted against each other. This is shown in Fig. 2, where sequences of equilibria corresponding to values of $\xi_0 = (0.9, 0.8, 0.7, 0.6)\pi$ are plotted. For a given ξ_0 an increase in η_* causes $\bar{\eta}$ to increase and α to decrease. However, there is a limit point for α and $\bar{\eta}$. That is, as η_* is allowed to become arbitrarily large α and $\bar{\eta}$ tend to constants depending only on ξ_0. This limit point can be determined analytically and will be discussed shortly.

From the practical point of view, one is not free to specify ξ_0 or η_*. Rather, one specifies the average density, $\bar{\eta}$, and deceleration rate, α. From Fig. 2 it is clear that as long as the values of $\bar{\eta}$ and α are chosen to lie below the curve joining the limiting points, an equilibrium with a specific ξ_0 and η_* exists. For points above this line no equilibrium exists with the given distribution. Thus, this line limits deceleration and density.

The precise energy dependence of the distribution function depends on the details of the way in which the particles were trapped. It will now be shown, however,

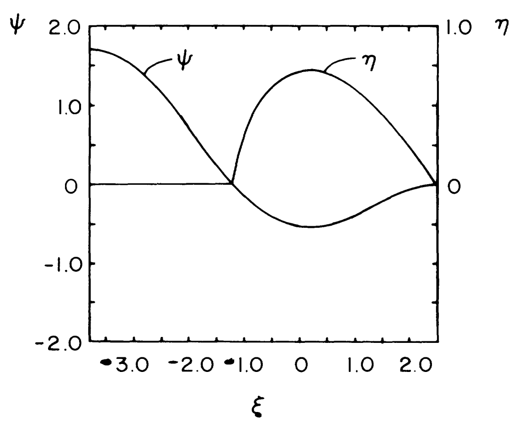

FIG. 1. Effective potential, ψ, and density, η, vs normalized distance for the following parameters: $\xi_0 = 0.8\pi$, $\alpha = 0.271$, and $\bar{\eta} = 0.299$.

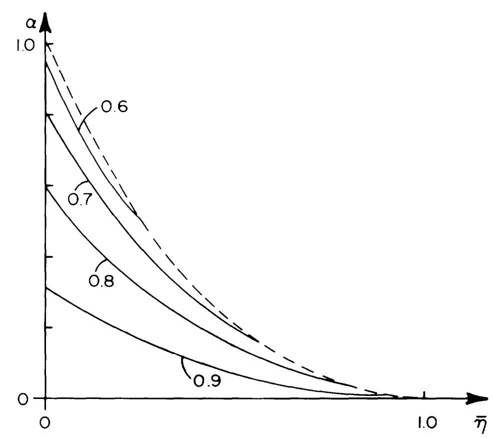

FIG. 2. Deceleration α vs average density $\bar{\eta}$ for $\xi_0 = (0.9, 0.8, 0.7, \text{and } 0.6)\pi$. The dashed curve is the limit obtained by solving of Eqs. (7), (8), and (9).

that the curve connecting the limit points is *independent of the details of the distribution function* so long as it is nonzero only for trapped particles.

The limit can be found by our noting that as one tries to force more and more particles into the well (for example, by increasing η_*) the well becomes shallower. Thus, if ξ_0 and ξ_1 are the two points where $\psi(\xi)=0$ and ψ is only slightly negative between these two points then in the limit of a vanishing well $\psi'(\xi_1)=\psi'(\xi_0)=0$. The limiting value of $\bar{\eta}$ is determined by the condition that the second derivative of ψ at $\xi=\xi_0$ be slightly negative,

$$\bar{\eta}_l=-\cos\xi_0. \tag{8}$$

The limiting value of α is then determined by the solving of Eq. (5) in the region between ξ_1 and $\xi_0-2\pi$, where $\eta(\psi)=0$ (no trapped particles):

$$\psi(\xi)=-\cos\xi+\bar{\eta}_l(\xi-\xi_1)^2/2+\sin\xi_1(\xi-\xi_1)+\cos\xi_1, \tag{9}$$

where the integration constants have been chosen so that $\psi(\xi_1)$ and $\psi'(\xi_1)$ vanish.

The quantity ξ_1 is determined by the boundary condition $\psi'(\xi_0-2\pi)=0$, and α_l follows from the value of $\psi(\xi_0-2\pi)$ by Eq. (7). The pairs α_l and η_l can be calculated by hand and appear in Fig. 2, confirming the results of the numerical integration of Eq. (5). Thus, the limiting curve applies to all distribution functions.

The assumptions made in obtaining these limiting values can be made more rigorous by our solving Eq. (5) asymptotically in the limit $\eta_*\rightarrow\infty$. The solution requires consideration of three separate regions of ξ between ξ_1 and ξ_0. Two of these regions are boundary layers near ξ_1 and ξ_0, respectively. The solutions in these regions confirm the assertions that $\psi'(\xi_1)\rightarrow 0$ and $\bar{\eta}+\cos\xi_0\rightarrow 0$ as $\eta_*\rightarrow\infty$. A detailed presentation of this analysis will be deferred to a future publication.

It is reasonable to ask what would happen if a beam with too high a current density were injected into the device. In this case only a fraction of the beam particles would be trapped. The nontrapped particles would then contribute to the density in Eq. (4). If one considers Eq. (4) far enough down the interaction length, then the nontrapped particles would have large values of ε (here ε measures energy in the ponderomotive wave frame). Because of the large value of energy these particles would produce a uniform density in z. In this case, Eq. (5) remains valid provided $\bar{\eta}$ is reinterpreted to be only the density of trapped electrons.

To assess the importance of this density limit consider the energy exchange between beam particles and the wave fields. As the particles are slowed down they give their energy to the field. Balancing the spatial rate of increase of wave energy with that lost by particles one finds

$$c\frac{\partial}{\partial z}\frac{E_r^2}{8\pi}=-\bar{n}mc^2\frac{d}{dt}\gamma_0=m\bar{n}\gamma_0^3v_0|\dot{v}_0|.$$

Expressing $|\dot{v}_0|$ and $\bar{n}$ in terms of the dimensionless parameters $\bar{\eta}$ and α one then obtains κ, the rate of exponentiation per wiggler period of the radiation field energy,

$$\frac{\kappa}{2\pi}=k_w^{-1}\frac{\partial}{\partial z}\ln E_r^2=\alpha\bar{\eta}\frac{\beta(1+\beta)}{2}\left[\frac{\Omega_w}{k_wv_0}\right]^2, \tag{10}$$

where $\Omega_w=qB_w/mc$ is the cyclotron frequency in the wiggler magnetic field. The product $\alpha\bar{\eta}$ has a maximum of 0.126 when $\alpha=0.409$ and $\bar{\eta}=0.308$. In practice one would have to operate below this maximum. Thus, Eq. (10) sets an upper (lower) limit on the rate of exponentiation (number of wiggler periods) in a free-electron–laser amplifier with tapering. It is interesting to note that with $\alpha\bar{\eta}=1$, Eq. (10) is within a numerical factor the same as obtained from a linear analysis[4] in the high-gain regime if one fixes the density to be at the transition from the Compton to Raman regimes. This is easily interpreted by our noting that at this transition the ponderomotive and electrostatic forces are equal (as they are in the present nonlinear calculation). Apparently, in the nonlinear regime the density cannot be pushed above this point.

Equation (10) can also be used to evaluate the performance of oscillators. If R is the power reflectivity of the output end of the oscillator cavity then one requires

$$R\exp(\kappa N)=1$$

in steady state where N is the number of wiggler periods.

In conclusion, the competition between the bunching force of the ponderomotive potential and the repulsive force of the self electrostatic potential in a tapered free-electron laser limits the density, deceleration rate, and electric field exponentiation rate. The maximum exponentiation rate possible depends only on beam velocity, wiggler field strength, and wiggler period, and is independent of the particle energy distribution function.

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