Nonlinear Oscillations in a Warm Plasma

E. Infeld[(a)]

Department of Applied Mathematics and Theoretical Physics, Cambridge CB3 9EW, United Kingdom

and

G. Rowlands

Department of Physics, Warwick University, Coventry CV4 7AL, United Kingdom

(Received 6 October 1986)

Nonlinear electron plasma oscillations in a warm plasma are studied by use of Lagrangean coordinates. Without recourse to amplitude expansion, the electron density is obtained as an explicit function of space and time. In the zero-temperature limit, a restriction on the previously found cold-plasma solutions is found.

PACS numbers: 52.35.Fp, 52.25.−b

The problem of nonlinear electron oscillations in a cold plasma was completely solved some time ago.[1-3] The method was based on the introduction of Lagrangean coordinates which follow the fluid motion. Calculations become very simple in these coordinates. Inversion to the fixed, Eulerian coordinates is given by an *implicit* relation. An expansion for the number density as a function of space and time is obtained without an amplitude expansion. A particular example such that inversion to Eulerian coordinates has been obtained *explicitly* can be found in Davidson's book,[4] Chapter 3. The important point, however, is that the general class of nonlinear plasma oscillations depends on two arbitrary functions of space. Thus any initial conditions can be considered. It has also been shown that traveling Bernstein-Greene-Kruskal modes,[5] depending on $x-Ut$ only, can be seen to follow from the above by suitable choice of these two functions.[6]

The availability of high-powered lasers has led to a renewed interest in highly nonlinear waves and in particular to the largest amplitude possible before the wave breaks.[7,8] This point will be considered at the end.

In this Letter, we extend the above considerations to include the effect of a finite temperature. Using the Lagrangean technique, followed by an additional coordinate transformation, we reduce the problem to the solution of one second-order partial differential equation. To solve this equation, we expand in the ratio of the Debye length λ_D characteristic of the plasma microstructure to that of the wave. This ratio is small in most physical situations and is in fact assumed small in the very derivation of the basic fluid equations used to describe the plasma. Therefore our expansion scheme leads to no loss of generality. There is, however, no amplitude expansion. A class of exact solutions is obtained in this context. An interesting result, obtained by consideration of the zero-temperature limit, is that the class of cold-plasma–wave solutions is drastically reduced. The two arbitrary functions mentioned above are replaced by two constants. This phenomenon will be further discussed at the end.

We consider an electron plasma described by the equations

$$\partial n_e/\partial t+\partial(n_eV_e)/\partial x=0, \tag{1}$$

$$\partial V_e/\partial t+V_e\,\partial V_e/\partial x=-E(-1/n_e)\partial P_e/\partial x, \tag{2}$$

$$\partial E/\partial x=1-n_e. \tag{3}$$

We additionally assume the adiabatic-isothermal pressure law

$$(\partial/\partial t+V_e\,\partial/\partial x)(P_e/n_e^\gamma)=0. \tag{4}$$

These equations are all in terms of dimensionless variables such that the electron density n_e is scaled to the constant neutralizing ion background density n_0. The length scale is the Debye length λ_D, and the time scale is the inverse of the plasma frequency ω_{pe} ($\lambda_D=P_0/2\omega_{pe}^2\times m_en_0$, $\omega_{pe}^2=4\pi n_0e^2/m_e$). The electron pressure is scaled to P_0, the electron pressure at points where the electron density is n_0. In what follows the subscript e will be dropped.

We now introduce Lagrangean coordinates (x,τ) which follow the electron fluid in terms of the Eulerian variables via the transformation

$$\tau=t,\quad \bar{x}=x-\int_0^\tau d\tau^1V(\bar{x},\tau^1). \tag{5}$$

Following Ref. 4, we find that (1)–(4) give, with suitable preparation of the initial pressure $p(\bar{x},0)=n^\gamma(\bar{x},0)$,

$$n(\bar{x},\tau)=n(\bar{x},0)\left[1+\int_0^\tau d\tau^1\frac{\partial}{\partial\bar{x}}V(\bar{x},\tau^1)\right]^{-1}, \tag{6}$$

and one differential-integral equation for V,

$$\frac{\partial^2V(\bar{x},\tau)}{\partial\tau^2}+V(\bar{x},\tau)=\frac{\gamma}{n(\bar{x},0)}\frac{\partial}{\partial\bar{x}}\frac{P(\bar{x},0)\partial V(\bar{x},\tau)/\partial\bar{x}}{[1+\int_0^\tau d\tau^1\,\partial V(\bar{x},\tau)/\partial\bar{x}]^{\gamma+1}}. \tag{7}$$

For a cold plasma P is zero and this equation obviously vindicates the introduction of a Lagrangean scheme. However, the rather complex full form of this equation undoubtedly detracted from further use of Lagrangean techniques in this context. Fortunately, however, it can be reduced to a differential equation. This is done by our differentiating (7) with respect to $\bar{x}$, integrating with respect to τ, and using (3) as an initial condition to give

$$(\partial^2/\partial\tau^2+1)\psi=1-\partial^2(\psi^{-\gamma})/\partial z^2 \qquad (8)$$

Here $\psi=n(\bar{x},\tau)^{-1}$ and the coordinate z is given by $z=\int^{\bar{x}}n(x,0)dx$. We note that if ψ is a function of $z-\mu\tau$ only, solutions correspond to traveling Bernstein-Greene-Kruskal waves. However, they are best investigated in Eulerian coordinates.[7]

To solve (8) we assume the space dependence to be gradual as compared with the scale of the plasma microstructure (wavelength/$\lambda_D\equiv 1/\epsilon\gg 1$). This allows us to introduce a coordinate stretching $z\to z/\epsilon$ and a multiple-time expansion,[4] $\tau_0,\tau_1,\tau_2,\ldots$, where $\tau_n=\epsilon^n\tau_0$. Also $\psi=\psi_0+\epsilon\psi_1+\ldots$. Now the lowest order form of (8) is

$$[\partial^2/\partial\tau_0^2+1]\psi_0=1, \qquad (9)$$

solved by

$$\psi_0=1+A(z,\tau_1)\cos\theta,\quad \theta=\tau_0+\phi(z,\tau_1). \qquad (10)$$

The first-order form is

$$(\partial^2/\partial\tau_0^2+1)\psi_1=2(\partial A/\partial\tau_1)\sin\theta+2A(\partial\phi/\partial\tau_1)\cos\theta-(\partial^2/\partial z^2)(1+A\cos\theta)^{-\gamma}. \qquad (11)$$

Conditions that no secular behavior arise are obtained by multiplying (11) first by $\sin\theta$ and then by $\cos\theta$ and integrating over $0<\theta<2\pi$. This procedure yields the two conditions

$$\frac{\partial A}{\partial\tau_1}=-I_\gamma^{-1}\frac{\partial}{\partial z}\left(I_\gamma^2\frac{\partial\phi}{\partial z}\right),\quad A\frac{\partial\phi}{\partial\tau_1}=\frac{\partial^2 I_\gamma}{\partial z^2}-I_\gamma\left(\frac{\partial\phi}{\partial z}\right)^2, \qquad (12)$$

where

$$I_\gamma(A)=\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta\cos\theta}{(1+A\cos\theta)^\gamma};\quad I_1=A^{-1}[1-(1-A^2)^{-1/2}],\quad I_3=-\tfrac{3}{2}A(1-A^2)^{-5/2}. \qquad (13)$$

As mentioned above, although obtained by an expansion technique, these equations are as accurate as (1)–(4). [The cold-plasma result is simply (10) with A and ϕ arbitrary functions of z.]

We now look for solutions to (12) such that A is a function of z only and $\phi=\xi\tau_1+\chi(z)$, $\xi>0$, and ξ is a constant frequency shift of order unity. Although this may seem restrictive, it does still include a large class of time-periodic oscillations, in particular the cold-plasma result. The $\xi=0$ case is not of interest, as it corresponds to infinite densities. Negative ξ will also be seen in what follows to be unphysical. Now (12) yields $d\chi/dz=c/I^2$, where c is a constant and

$$A\xi=d^2I/dz^2-c^2/I^3. \qquad (14)$$

This equation can be solved by our writing $dA/dz=B(A)$, $B^2=R/(dI/dA)^2$, leading to

$$d(r+c^2/I^2)/dA=2\xi A\,dI/dA, \qquad (15)$$

to give

$$R(A)=b^2-2\xi\{\ln(\tfrac{1}{2}[1+(1-A^2)^{1/2}])-[1-(1-A^2)^{-1/2}]\}-c^2/I_1^2(A),\quad \gamma=1,$$

$$R(A)=b^2-\xi[3(1-A^2)^{-5/2}-4(1-A^2)^{-3/2}+1]-c^2/I_3^2(A),\quad \gamma=3,$$

where b is an integration constant.

For $\xi>0$, $0<c^2<c_{\max}^2$, $0<b^2<b_{\max}^2$, $R(A)$ is nonnegative between two roots of R, A_1 and A_2, such that $0<A_1<A<A_2<1$. For $c=0$ the condition becomes $-1<-A_1<A<A_1<1$. For $\xi<0$ there is no bounded solution. This is a nonlinear generalization of the fact that for a linear plasma wave, the frequency-squared shift is always positive. Apart from ξ, the solution is given in terms of two constants b and c.

The relationship between x and A can be found from the above equations to be

$$\bar{x}=\int^{\bar{x}}dx=-\int_{A_1}^{A}dA\,\frac{dI/dA}{[R(A)]^{1/2}}(1+A\cos\chi),\quad A_1<A<A_2. \qquad (16)$$

We now concentrate on the case $c=0$, so that χ is a constant, leaving a more general discussion to a fuller version of this work.

If we use (5) and express V in terms of ψ, simple integration leads to the expression

$$x=\bar{x}-(1/\xi)[R(A)]^{1/2}[\cos(\hat{\omega}t+\chi)-\cos\chi],\quad \hat{\omega}=1+\xi. \qquad (17)$$

Combining (16) and (17) we obtain

$$\xi^{1/2}x = -\int_{A_1}^{A} dA \frac{dI/dA}{[\bar{R}(A)]^{1/2}} - [\bar{R}(A)]^{1/2}\cos(\hat{\omega}t+\chi), \quad \bar{R}=R/\xi. \tag{18}$$

Importantly the Lagrangean variables have now been eliminated and Eq. (18) can be inverted to give A as a function of x and t. The form of A so found can be substituted into

$$n(x,t) = [1 + A(x,t)\cos(\hat{\omega}t+\chi)]^{-1} \tag{19}$$

to give the spatial and temporal dependence of the electron-number density.

To illustrate the salient features of this solution, we approximate the functions of A that appear in (18) so as to give an explicit form of $n(x,t)$. The integrand is approximated by the parabolic fit $\alpha_1(1-A^2/A_1^2)^{-1/2}$, which has the correct behavior at the poles and where α_1 is chosen such that the wave number k is the same as the exact value so that

$$k \equiv \pi\xi^{1/2}\int_0^{A_1} \frac{dA\, dI/dA}{[\bar{R}(A)]^{1/2}} = \frac{\xi^{1/2}}{\alpha_1 A_1}. \tag{20}$$

Similarly, we approximate $R(A)$ by $\alpha_2(1-A^2/A_1^2)$ with α_2 chosen to ensure density conservation. Our approximations are good for all b, and do not assume small amplitudes. If we introduce $A/A_1=\cos\eta$, Eq. (18) reduces to

$$kx = \eta + A_1\cos\eta\cos(\hat{\omega}t+\chi). \tag{21}$$

This equation can be solved, giving finally

$$\frac{1}{n} = 1 + 2A_1\cos(\hat{\omega}t+\chi)\left\{\frac{A_1}{4} + \sum_{n=1}^{\infty}\frac{1}{n}J_n'(-nA_1\cos(\hat{\omega}t+\chi))\cos(nkx)\right\}. \tag{22}$$

This gives the space and time variation of the number density which for A_1 near its maximum value of unity can have steep spatial variation, though for $\hat{\omega}t+\chi=(2m+1)\pi/2$, m any integer, n is spatially uniform.

One can associate the time variation with a fundamental frequency $\hat{\omega}$ ($\equiv 1+\xi$). If we use (20) to eliminate ξ and revert to physical variables, this may be written in the form

$$\omega^2 = \omega_p^2[1 + k^2\lambda_D^2 G(A_1,\gamma)], \tag{23}$$

where $G(A_1,\gamma)\to 1$ for $A_1\to 0$, thus reproducing the well-known linear dispersion relation [a term of order $k^4\lambda_D^4$ has been neglected but as stressed above this is consistent with the range of validity of Eqs. (1) to (4)].

Although the above solution (22) looks somewhat like that found for the cold-plasma case (see Ref. 4, p. 40), it is in fact fundamentally different. In the cold-plasma case, the disturbance at $t=0$ can be arbitrary, but was specified to be of the form $n=1+\Delta\cos kx$. In the present, warm-plasma case, the shape at $t=0$ results from the calculation and it is not a cosine as may be seen by our putting $t=0$ in (22). Whereas in the cold-plasma case, one is free to shape the wave initially, A and χ being arbitrary functions of space, in the warm-plasma case we can only specify the amplitude and wavelength (A_1 and k). This drastic reduction in the class of possible solutions is not so surprising if we look at linear wave theory. In such a theory the dispersion relation is $\omega^2=\omega_p^2(1+k^2\lambda_D^2)$, and gives a unique value of k^2 for a given frequency ω. Therefore only one mode can be excited. In contrast to this, for a cold plasma $\omega^2=\omega_p^2$ and any combination of spatial modes is allowed. This difference is reflected in the nonlinear regime as found in the present analysis, arbitrary functions in the cold-plasma case being replaced by constants for the warm plasma. Mathematically this is seen in Eq. (18), which admits arbitrary space dependence for $\epsilon\equiv 0$, but not for $\epsilon\to 0$.

In summary it has been shown that thermal effects can be incorporated into a Lagrangean-based theory, expressed in the form of a simple differential Eq. (8). Above all we have seen how, at least for a particular case, the well-known concept of a standing wave extends into nonlinear plasma physics. In particular the concept of a dispersion relation can still have meaning for all amplitudes.

The class of solutions of the basic Eq. (12) has been restricted to those where $\partial A/\partial\tau_1=0$. Although this class includes the cold-plasma case, its full implications are not fully understood. On the other hand, the removal of the condition $c=0$ simply leads to more complicated forms for $n(x,t)$ but not to any quantitative differences.

A number of authors[7,8] have used various nonlinear theories to estimate the maximum amplitude of the electric field, E_{max}, allowed before the wave breaks. In the present calculation the condition is, from (19), simply $A(x,t)=1$ for some x and t, and since the maximum value of $A(x,t)$ is A_1 the condition is simply $A_1=1$. It will be noted that this value is independent of the thermal velocity V_T. If one carries out the analysis to next order and evaluates ψ_1, then a correction of order ϵ ($\propto V_T^2$) is obtained. In two distinct problems considered previously, that of forced oscillation[7,8] and of Bernstein-Greene-Kruskal waves in one-dimensional "water-bag plasma," E_{max} was found to be of the form of a constant minus a term proportional to $V_T^{1/2}$, for small V_T. Thus comparatively speaking, the standing waves discussed in this Letter have E_{max} almost temperature independent.

This weak V_T dependence for standing waves seems more natural when one considers how energy scales. The increment in the electric field energy, $(\Delta E_m)^2/8\pi$, when we go from zero to small but finite V_T, scales as V_T^2. The reason why this is not found in the two cases mentioned above is that the energy considerations are more complicated—for forced oscillations by the presence of the laser field which drives the wave, while for Bernstein-Greene-Kruskal waves by the energy contribution associated with the constant phase velocity of the wave. (Incidentally, the $V_T^{1/2}$ dependence common to the two cases is not universal. For an "isothermal" plasma the correction is proportional to V_T as can be derived from the result of Infeld and Rowlands.[9]) Thus we conclude that the temperature dependence of $E_{\max}$ depends critically on the experimental arrangement and that in particular for standing nonlinear waves the dependence on V_T is weak.

[(a)]On leave from Institute of Nuclear Studies, Warsaw 00-681, Hoza 69 Poland.

[1]P. A. Sturrock, Proc. Roy. Soc. London, Ser. A **242**, 277 (1957).

[2]J. M. Dawson, Phys. Rev. **113**, 383 (1959).

[3]R. C. Davidson and P. P. Schram, Nucl. Fusion **8**, 183 (1968).

[4]R. C. Davidson, *Methods in Nonlinear Plasma Theory* (Academic, New York, 1972).

[5]I. B. Bernstein, J. M. Greene, and M. D. Kruskal, Phys. Rev. **108**, 546 (1957).

[6]J. Albritton and G. Rowlands, Nucl. Fusion **15**, 1199 (1975).

[7]W. L. Kruer, Phys. Fluids **22**, 1111 (1979).

[8]T. Speziale and P. J. Catto, Phys. Fluids **20**, 990 (1977).

[9]E. Infeld and G. Rowlands, J. Phys. A **12**, 2255 (1979).