New Cellular Automaton Model for Magnetohydrodynamics

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(Received 2 February 1987)

A new type of two-dimensional cellular automation method is introduced for computation of magnetohydrodynamic fluid systems. Particle population is described by a 36-component tensor referred to a hexagonal lattice. By appropriate choice of the coefficients that control the modified streaming algorithm and the definition of the macroscopic fields, it is possible to compute both Lorentz-force and magnetic-induction effects. The method is local in the microscopic space and therefore suited to massively parallel computations.

PACS numbers: 52.30.−q, 47.10.+g, 47.65.+a, 52.65.+z

The recent development of a hexagonal lattice-gas (HLG) model[1] for two-dimensional hydrodynamics has led to a considerable level of interest in the use of cellular automata[2] (CA) for the study of fluid and fluidlike physical systems.[3] CA fluid models may offer significant computational advantages[1] and provide insights into the relationship between macroscopic physics and the nature of the microphysical world.[1-3] Recently, Montgomery and Doolen[4] introduced a two-dimensional magnetohydrodynamics (MHD) model that makes use of both microscopic cellular-automata and macroscopic finite-difference methods. Their model departs from the usual notion[2] of cellular automata by a nonlocal computation of the Lorentz force, involving spatial differences of the coarse-grained magnetic potential. It has been suggested[5] that nonlocal features of CA models of MHD and other plasma systems may be inescapable in view of the nonlocal physics that pervades the approximations leading to MHD. In the context of our own investigations[6] of a passive scalar CA model and its generalization to MHD we concluded, in accordance with Ref. 5, that nonlocal features are inevitable for this type of MHD model, but we attribute the nonlocality to the formulation in terms of the vector potential. In this Letter we present an alternative formulation of MHD cellular automation in which the microscopic dynamical rules are completely local in both time and space. The new method would appear to be well suited to large-scale parallel computation.

The system of two-dimensional incompressible MHD equations[7,8] for which we develop a CA model may be written as

$$\rho(\partial v/\partial t+\mathbf{v}\cdot\nabla\mathbf{v}) = -\nabla p(\nabla\times\mathbf{B})\times\mathbf{B}+\nu\nabla^2\mathbf{v}, \tag{1}$$

$$\partial\mathbf{B}/\partial t+\mathbf{v}\cdot\nabla\mathbf{B}=\mathbf{B}\cdot\nabla\mathbf{v}+\mu\nabla^2\mathbf{B}, \tag{2}$$

where $\mathbf{B}$, $\mathbf{v}$, p ρ, μ, and ν are the magnetic field, velocity field, pressure, mass density, resistivity, and viscosity, respectively. For incompressible flow $\nabla\cdot\mathbf{v}=0$ and $\rho=$const, while the pressure is determined from the Poisson equation that results from computation of the divergence of (1). In the relevant two-dimensional (x,y) geometry $\mathbf{v}$ and $\mathbf{B}$ lie in the x-y plane and depend only on those coordinates and time. The magnetic potential A_z is related to $\mathbf{B}$ by $\mathbf{B}=\nabla\times A_z\hat{\mathbf{e}}_z$ where $\hat{\mathbf{z}}=\hat{\mathbf{x}}\times\hat{\mathbf{y}}$.

The possibility that all the nonlinearities in (1) and (2) might be handled locally by a CA model may be easily motivated by consideration of the structure of (1) and (2), with neglect of pressure and dissipation, in terms of the Elsässer[9] variables $\mathbf{z}^{\pm}=\mathbf{v}\pm\mathbf{B}/\rho^{1/2}$. From $\partial\mathbf{z}^{\pm}/\partial t\approx-\mathbf{z}^{\mp}\cdot\nabla\mathbf{z}^{\pm}$ it is easily seen that the relevant nonlinearities, including Lorentz force, are nonlocal only in an appropriately generalized advective sense. Ordinary advection due to the velocity field can be adequately treated in both hydrodynamic[1] and vector-potential–based MHD[4-6] CA models. The Lorentz force $(\nabla\times\mathbf{B})\times\mathbf{B}$ written as $-\nabla A_z\nabla^2A_z$ cannot be dealt with in this way because it involves a nonlinear product of a quantity having components of the gradient that are not parallel to $\mathbf{v}$, with a quantity having a second derivative which must involve information from neighboring cells. On the basis of the Elsässer-variable argument it would seem necessary to treat $\mathbf{B}$ on more nearly equal footing with $\mathbf{v}$ to achieve a local MHD CA model.

The basis of the present model is a modified streaming algorithm for particles moving on a hexagonal grid in which each particle occupies a state labeled by two vectors, $\hat{\mathbf{e}}_a$ and $\hat{\mathbf{e}}_b$, where $\hat{\mathbf{e}}_a=(\cos2\pi a/6,\sin2\pi b/6)$, $\hat{\mathbf{e}}_b=(\cos2\pi b/6,\sin2\pi b/6)$ and both a and b run from 1 to 6. No more than one particle in each cell may occupy a state with a specified a and b, so that at most 36 particles may simultaneously reside in a cell. Letting N_a^b ($=0$ or 1) denote the occupation number at a certain location, we define $f_a^b\equiv\langle N_a^b\rangle$ to be the ensemble-averaged particle distribution. At each CA time level, streaming, by which we mean the noncollisional component of particle motion, consists of motion to the adjacent cell in the direction $\hat{\mathbf{e}}_a$ with probability $1-|P_{ab}|$. Alternatively, with probability $|P_{ab}|$ the particle moves to the adjacent cell in the direction $\hat{\mathbf{e}}_bP_{ab}/|P_{ab}|$. This leads to a kinetic equation[3] for the tensor particle distribution f_a^b,

$$\partial f_a^b(\mathbf{x},t)/\partial t=-\{(1-|P_{ab}|)\hat{\mathbf{e}}_a+P_{ab}\hat{\mathbf{e}}_b\}\cdot\nabla f_a^b(\mathbf{x},t)+\Omega_{ab}, \tag{3}$$

where Ω_{ab} represents the effect of all collisions that modify f_a^b.

The macroscopic number density, fluid velocity field, and magnetic field will be designated as n, $\mathbf{v}$, and $\mathbf{B}$, respectively, and will be related to the microstate by the relations

$$n = \sum_{a,b} f_a^b, \tag{4}$$

$$n\mathbf{v} = \sum_{a,b} \{(1-|P_{ab}|)\hat{\mathbf{e}}_a + P_{ab}\hat{\mathbf{e}}_b\} f_a^b, \tag{5}$$

$$n\mathbf{B} = \sum_{a,b} \{Q_{ab}\hat{\mathbf{e}}_b + R_{ab}\hat{\mathbf{e}}_a\} f_a^b, \tag{6}$$

where the 6×6 matrices $\mathbf{P}$, $\mathbf{Q}$, and $\mathbf{R}$ are as yet undetermined constants that must be selected on theoretical grounds to give the desired physical behavior of MHD. By requiring that the behavior of the system be locally invariant under both proper and improper rotations, we conclude that $\mathbf{P}$, $\mathbf{Q}$, and $\mathbf{R}$ must be circulant[3] matrices, and that P_{ab}, Q_{ab}, and R_{ab} depend only on $|a-b|$. There remain twelve coefficients in these matrices that must be selected to specify the streaming behavior of the model, from which all the nonlinearities of MHD will emerge.

Further simplification is obtained by consideration of symmetries of the Lorentz force that are implied by the structure of (1) and (2). Notice that the Lorentz force is unchanged by the transformation $\mathbf{B}\rightarrow -\mathbf{B}$. We wish to have this macroscopic transformation correspond to the microscopic transformation $\hat{\mathbf{e}}_b \rightarrow -\hat{\mathbf{e}}_b$, so that $\hat{\mathbf{e}}_b$ will act as the magnetic quantum in the same way that $\hat{\mathbf{e}}_a$ controls undeflected momentum transport. To achieve this, we require that $\mathbf{B}\rightarrow -\mathbf{B}$ everywhere whenever $\hat{\mathbf{e}}_b \rightarrow -\hat{\mathbf{e}}_b$, but that neither $\mathbf{v}$ nor $\hat{\mathbf{e}}_a$ is affected. Upon consideration of (5) and (6), one can see that this requires $P_{ab} \equiv -P_{ab+3}$ to ensure that the velocity is unchanged, while $Q_{ab}\equiv Q_{ab+3}$ and $R_{ab}\equiv -R_{ab+3}$. Combining this with the rotational and reflection symmetries, we are left with just six independent coefficients, chosen to be P_{aa}, P_{aa+1}, Q_{aa}, Q_{aa+1}, R_{aa}, and R_{aa+1}. With these choices the transformation $\hat{\mathbf{e}}_b \rightarrow -\hat{\mathbf{e}}_b$ will cause $\mathbf{B}\rightarrow -\mathbf{B}$ without changing $\mathbf{v}$ while the transformation $\hat{\mathbf{e}}_a \rightarrow -\hat{\mathbf{e}}_a$ will lead to $\mathbf{v}\rightarrow -\mathbf{v}$ without modifying $\mathbf{B}$. This choice of symmetry also accounts for physically correct behavior of fluid elements in simple macroscopic field geometries. Bidirectional streaming allows the average trajectory of a particle to vary relative to $\hat{\mathbf{e}}_a$ and $\hat{\mathbf{e}}_b$. Consider for the moment the most probable $\hat{\mathbf{e}}_b$ to be a good estimate of the local macroscopic $\mathbf{B}$. Particles streaming across a simple sheared $\mathbf{B}$ can easily be understood to experience a deflecting force that is qualitatively consistent with the Lorentz force since deflection will be towards the most probable $\hat{\mathbf{e}}_b$ when the angle $\theta = \cos^{-1}(\hat{\mathbf{e}}_a\cdot\hat{\mathbf{e}}_b)$ is acute or towards $-\hat{\mathbf{e}}_b$ when the angle is obtuse. Moreover, the effect of magnetic pressure is also correctly accounted for by acceleration of particles towards the weaker magnetic field region where the most probable $\hat{\mathbf{e}}_b$ will be encountered less frequently.

Following previous CA fluid-model developments[1,3,4] collision rules are adopted to randomize the microscopic state while preserving macroscopic quantities that are found to be necessary to give correct ensemble averaged behavior. Essential to the approach are inequalities between collisional and macroscopic time and length scales that allow the local microstate to be treated as near to equilibrium. The same approach is adopted here, with the requirement that the collision term, Ω_{ab} in Eq. (3), satisfies $\sum_{a,b}\Omega_{ab}=0$,

$$\sum_{a,b} \{(1-|P_{ab}|)\hat{\mathbf{e}}_a + P_{ab}\hat{\mathbf{e}}_b\}\,\Omega_{ab} = 0,$$

and $\sum_{a,b}\{_{ab}\hat{\mathbf{e}}_b + R_{ab}\hat{\mathbf{e}}_a\}\,\Omega_{ab}=0$. These represent conservation of particle number, the momentum $n\mathbf{v}$, and the density-weighted magnetic field $n\mathbf{B}$. A large number of possible collision rules satisfy these requirements, many of them straightforward extensions of hydrodynamic CA collisions.[1,3]

Without further approximation, and with no assumptions about the form of the distribution function, the equation of particle transport is obtained from the kinetic equation (3) by summation over a and b, and then use of conservation of particles by collisions and Eqs. (4) and (5). This leads to

$$\partial n/\partial t + \nabla\cdot(n\mathbf{v}) = 0. \tag{7}$$

This is the ordinary fluid-continuity equation.

In order to deduce transport equations for $\mathbf{v}$ and $\mathbf{B}$, which will be cast in the form of Eqs. (1) and (2), it is necessary to rely on collisions to produce a local equilibrium state. In the lowest-order Chapman-Enskog expansion,[1,3,4] the collisions lead to a local Fermi-Dirac equilibrium distribution

$$f_a^b(\text{equil}) = 1/[1+\exp(\alpha + \boldsymbol{\beta}\cdot\hat{\mathbf{e}}_a + \boldsymbol{\eta}\cdot\hat{\mathbf{e}}_b)], \tag{8}$$

where α, β, and η are obtained by the definitions of n, $\mathbf{v}$, and $\mathbf{B}$ in (4)-(6).

The momentum equation is obtained by our multiplying both sides of (3) by $\{(1-|P_{ab}|)\hat{\mathbf{e}}_a + P_{ab}\hat{\mathbf{e}}_b\}$, summing over a and b, and using Eqs. (3)-(5). To proceed we consider the expansion of (8) in the limit $|\mathbf{v}|\ll 1$ and $|\mathbf{B}|\ll 1$. After some tedious algebra, which is facilitated by use of the symmetries adopted above, we arrive at

$$\partial(n\mathbf{v})/\partial t = -C_1\nabla n/6 - C_2\nabla\cdot[nG(n)\mathbf{v}\mathbf{v}] + C_3\nabla\cdot[nG(n)\mathbf{B}\mathbf{B}] + C_4\nabla[nG(n)\mathbf{v}^2] + C_5\nabla[nG(n)\mathbf{B}^2]. \tag{9}$$

In Eq. (9), $G(n)\equiv(18-n)/(36-n)$ and C_1, C_2, C_3, C_4, and C_5 are rational functions of the six independent streaming coefficients in the matrices $\mathbf{P}$, $\mathbf{Q}$, and $\mathbf{R}$. The above relation contains terms of the same general form as the momentum equation obtained in other fluid CA models,[1,4] except that correctly structured terms involving the magnetic field also appear.

Similarly, an equation of the same general form as the induction equation (2) is obtained after multiplication of (3) by $Q_{ab}\hat{\mathbf{e}}_b + R_{ab}\hat{\mathbf{e}}_b$ and summation over all a and b. The result is

$$\partial(n\mathbf{B})/\partial t = -D_1\nabla\cdot[nG(n)\mathbf{vB}] + D_2\nabla\cdot[nG(n)\mathbf{Bv}] + D_3\nabla[n\mathbf{v}\cdot\mathbf{B}], \quad (10)$$

where D_1, D_2, and D_3 depend only on the streaming coefficients.

To arrive at a CA model for MHD a number of restrictions must be placed on the coefficients in (9) and (10). For example, the last term in (10) must be eliminated, since it does not appear in (2) and will generate nonsolenoidal magnetic fields; thus $D_3=0$ must be enforced. For nonnegative pressure, $C_1>0$ is required. Furthermore, for consistency with (1) and (2), we must choose $C_2=C_3=D_1=D_2>0$. Along with the constraints that $|P_{ab}|<1$ (to allow a probabilistic interpretation of the modified streaming) and the vanishing of D_3, we arrive at four constraint equations and four inequalities that restrict allowed values of streaming coefficients P_{aa}, P_{aa+1}, Q_{aa}, Q_{aa+1}, R_{aa}, and R_{aa+1}. We have found numerical solutions to the constraints that indicate the existence of continuous ranges of allowed parameters, all of which have $P_{aa}<0$. One solution is $P_{aa}=-\frac{1}{3}$, $P_{aa+1}=+\frac{2}{9}$, $Q_{aa}=\frac{1}{2}$, $Q_{aa+1}\approx 0.065$, $R_{aa}=0$, and $R_{aa+1}\approx -0.232$.

Having solved the constraint equations the final result for the zeroth-order macroscopic behavior of the MHD CA model is

$$\partial(n\mathbf{v})/\partial t = -C_2\nabla\cdot[nG(n)(\mathbf{vv}-\mathbf{BB})] - \nabla[C_1 n/6 - nG(n)(C_4\mathbf{v}^2 + C_5\mathbf{B}^2)], \quad (11)$$

$$\partial(n\mathbf{B})/\partial t = -C_2\nabla\cdot[nG(n)(\mathbf{vB}-\mathbf{Bv})]. \quad (12)$$

For the above solution we have found $C_1=1.77$, $C_2=1.09$, $C_4=-C_5=C_2/2$. For $|\mathbf{v}|\ll 1$ and $|\mathbf{B}|\ll 1$, corresponding to the low-Mach-number flow limit, the equation of state gives a lowest-order relation between pressure and density of the form $p=C_1 n/6$, with the additional anisotropic effects of order $\mathbf{v}^2$ and $\mathbf{B}^2$. In the same limit, the density will exhibit only small fluctuations about a uniform constant value[1,3] so that the factors of n in various terms cancel and the factor $C_2G(n)$ may be used to rescale the relationship between microscopic and macroscopic time.[1] This leads to a set of dynamical equations almost identical to incompressible MHD. All numerical solutions to the constraints that we have found have the property that $C_4=-C_5=C_2/2$, leading to an exact representation of the Lorentz force in (11).

There are a number of additional issues important to the development of the MHD CA model that warrant brief mention here; a detailed description of the model will be forthcoming.[10] First, the allowed collisions are closely related to those in the six-state hexagonal lattice gas (HLG),[1,3] and always involve sets of particles with zero net $\mathbf{v}$ and $\mathbf{B}$. At very low densities $n\ll 1$ collisions will be infrequent and the collisional mean free path may be unacceptably large. Fortunately there appears to be no restriction on running the model at higher density, except that $n<18$ [so that $G(n)>0$]. However, certain manipulations in the lattice kinetic theory may be difficult to justify[11] for high densities. The key restriction on interpreting (11) and (12) as a model of MHD is that the density be very nearly uniform and the flow therefore incompressible. This approximation is favored both by substantial inequality between the characteristic microscopic and macroscopic length and by small amplitudes of the macroscopic fields, equivalent to the low-Mach-number[1] condition for the HLG. This latter condition should be no more restrictive here than for the HLG since the sound speed for the present model is $c_s\approx (C_1/6)^{1/2}$. Moreover, small departures from incompressibility should behave properly as sound waves in the nearly quiescent state by the same reasoning used in the HLG case.[3] We are currently investigating[10] the behavior of MHD Alfvén and magnetosonic[7] waves. Another issue of importance is the requirement that $\nabla\cdot\mathbf{B}=0$. The choice of streaming coefficients leading to $D_3\equiv 0$ eliminated the most seriously offending term in (10), but it is not possible to eliminate $\nabla\cdot\mathbf{B}$ exactly. On the other hand, from the divergence of (12), we find that $\partial\nabla\cdot\mathbf{B}/\partial t$ is at most of the order of the density inhomogeneities, presumably Mach number squared. This is no more restrictive than the low-Mach-number requirement for incompressibility. Moreover, diffusion decreases $\nabla\cdot\mathbf{B}$. It will be important, however, to initialize the model with a magnetic field that is as nearly divergence-free as possible. The viscosity[1,3,4] and magnetic diffusivity[6] in this model have not yet been computed in view of the number of degrees of freedom involved. Nevertheless, we have found that the first-order Chapman-Enskog expansion gives the correct structure of the diffusion terms, i.e., they are proportional to $\nabla^2\mathbf{v}$ in the momentum equation and $\nabla^2\mathbf{B}$ in the induction equation.

In summary we find that field-line stretching and Lorentz forces can be incorporated into a local CA model by the introduction of a microscopic bidirectional streaming procedure. The allowed particle states on the hexagonal lattice are labeled by two vectors and the direction of particle motion is selected according to prescribed streaming coefficients, leading to a modified kinetic equation involving a tensor particle distribution. The definition of the macroscopic velocity and magnetic fields also depend on the streaming coefficients. With proper choice of the coefficient matrices **P**, **Q**, and **R**, MHD is recovered for low fluid speed and low magnetic field strength. The success of the model is associated with the choice of symmetries for the streaming coefficients, which supports the notion[2] that simplified microscopic models may exhibit physically meaningful macroscopic behavior when the microscopic conservation laws and symmetries are correctly constructed. It is also likely that systems of equations other than MHD might be modeled by CA methods in a similar fashion. The idea of multidirectional streaming has some precedent in the recent CA model of Boghosian and Levermore[12] for the one-dimensional Burgers equation. The present model is well suited for parallel computation on machines such as the massively parallel processor because the microscopic behavior is independent of the macroscopic state. Numerical experimentation and comparison with standard computational methods will be needed to assess the potential utility[1] and possible limitations[13] of this CA MHD model.

Useful discussions with Professor David Montgomery, Dr. John Dorband, and Larry Klein are gratefully acknowledged. This research supported by the National Science Foundation under Grant No. ATM-8609740, and by NASA through the Director's Office and the Solar Terrestrial Theory Program at Goddard Space Flight Center.

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