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Power Spectra of Strange Attractors near Homoclinic Orbits

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Assuming that the chaotic time history of a single variable in a differential equation possessing a strange attractor can be represented as the random superposition of deterministic "structures," we predict the power spectral density. We justify the assumption for perturbations of nonlinear Hamiltonian oscillators and compare our predictions with computations on versions of Duffing's equation.

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Dynamical systems possessing strange attractors have been proposed as models for a number of physical processes which display erratic temporal behavior.[1-3] As well as the abstract theory,[4] there are analytical techniques for the study of global behavior in *specific* systems. In particular, Melnikov's method[2,5] detects transverse homoclinic points in differential equations which are small perturbations of integrable (Hamiltonian) systems. This, with the Smale-Birkhoff homoclinic theorem,[2,4] implies the existence of chaotic motions among the solutions of the equation in question: qualitative information. In contrast, here we propose a method which provides quantitative statistical measures of solutions: We compute power spectra of chaotic motions which are perturbations of homoclinic orbits. Our approach relies on the existence of global homoclinic structures, verifiable by Melnikov theory, and derives from the notion of coherent structures in turbulence theory.[6] It has been proposed before in connection with differential equations,[7] although this earlier work does not provide *a priori* estimates from the unperturbed equations, as does ours. Spectral estimates have also been proposed in connection with the scaling properties of period doubling and halving[8] and with intermittency.[9]

For simplicity, we focus on perturbations of the Duffing equation

$$\dot{x} = y,$$
$$\dot{y} = x - x^3 + \epsilon(\gamma\cos\nu t - \delta y + \beta x^2 y), \quad 0 \le \epsilon \ll 1, \qquad (1)$$

although our ideas are more generally applicable. For $\epsilon=0$ the unperturbed phase plane of (1) has a pair of homoclinic orbits to the saddle point $(x,y)=(0,0)$: $\Gamma_{-1} \cup \Gamma_{+1}$, solutions in which may be written

$$x_+(t) := s(t) = \sqrt{2}\,\mathrm{sech}\,t, \quad x_-(t) = -s(t) \qquad (2)$$

(Fig. 1). The Melnikov method concerns orbits which remain near $\Gamma_{\pm 1}$ when $\epsilon \neq 0$ and involves computation of

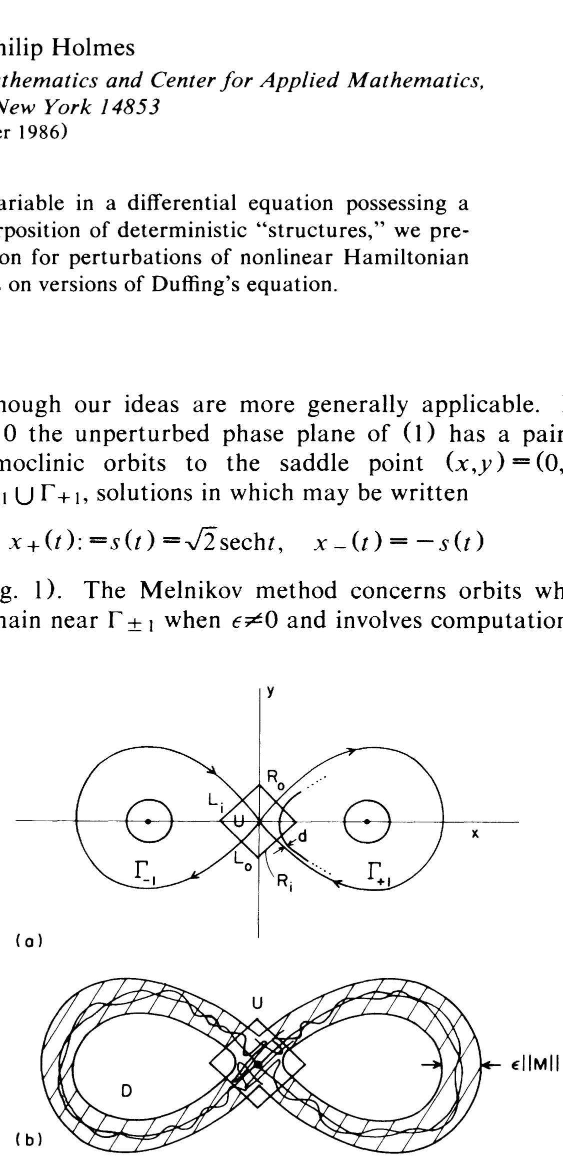

FIG. 1. The Duffing oscillator: (a) Unperturbed phase plane; (b) perturbed Poincaré map showing transverse homoclinic points and trapping region D.

the function

$$M(\theta)=\int_{-\infty}^{\infty}\dot{s}(t)[\gamma\cos\nu(t+\theta)-\delta\dot{s}(t)+\beta s^2(t)\dot{s}(t)]dt=\sqrt{2}\gamma\pi\nu\,\mathrm{sech}(\tfrac{1}{2}\pi\nu)\sin\nu\theta-4\delta/3+16\beta/15 \quad (3)$$

[for Eq. (1)]. If M has simple zeros, then, *for sufficiently small* ϵ, chaotic motions exist near $\Gamma_{\pm 1}$.[2,5] If we assign a bi-infinite sequence $\mathbf{a}(z)=\{a_j\}_{-\infty}^{\infty}$ to each initial condition $z=(x(0),y(0))$ which reflects the behavior of the solution $x(t)$ based at z in that $a_k=0$ or 1 depending on whether the jth maximum of $|x(t)|$ occurs near Γ_{+1} ($x>0$) or Γ_{-1} ($x<0$), then *every possible sequence is realized.* Moreover, 0 and 1 have equal probabilities and the process has no memory. Thus, there are solutions which appear indistinguishable from random processes such as coin tossing.[2,4,10]

These chaotic motions do not necessarily constitute a strange attractor[2]: Orbits may escape from the chaotic set and approach stable periodic motions. The homoclinic tangencies predicted by Melnikov analysis *guarantee* that stable periodic orbits exist for residual subsets of parameter values.[10,11] However, these orbits correspond to long periods and are not typically observed. In the following we *assume* that small (numerical) errors destabilize any such sinks. This assumption is essential: There is no proof yet of the existence of a true strange attractor (dense orbit) for a specific differential equation.

If $\delta\approx 4\beta/5>0$, almost all orbits starting in some disk are attracted to a neighborhood D of $\Gamma_{\pm 1}$ of width $\epsilon\gamma\nu\,\mathrm{sech}(\frac{1}{2}\pi\nu)$; see Fig. 1.[10] This explains our choice of perturbation: We wish to control the solutions as far as possible (but see below). A typical chaotic solution $x(t)$ can thus be approximated by

$$x(t)=\sum_{j=-\infty}^{\infty}(-1)^{a_j}s(t-T_j), \quad (4)$$

where $a_j\in\{0,1\}$ is the symbol described above, $s(t)$ is the unperturbed homoclinic loop Γ_{+1} [Eq. (2)], and T_j is the time at which the jth maximum in $|x(t)|$ occurs. Moreover, it is reasonable to suppose that $a_j\in\{0,1\}$ and $T_j\in\mathbb{R}$ are random variables.

Multiplying $x(t)$ by a "window" function $g_L(t)$ of compact support [$g_L(t)=1$, $|t|<L$; $g_L(t)=0$, $|t|>L$], we rewrite the (integrable) windowed solution $x_L(t)$ as a convolution integral $x_L(t)$:

$$\int_{-\infty}^{\infty}a_L(t'-t)s(t')dt',$$

where

$$a_L(\tau)=g_L(\tau)\textstyle\sum_j(-1)^{a_j}\delta(\tau+T_j)$$

is a (finite) random sequence of delta functions (shot noise). The Fourier transform of $x_L(t)$ is then the product $\hat{a}_L(f)\hat{s}(f)$ of the transforms[12] and the power spectral density of $x(t)$ is

$$E_x(f)=\lim_{L\to\infty}(1/2L)|\hat{x}_L(f)|^2=\lim_{L\to\infty}(1/2L)|\hat{a}_L(f)|^2|\hat{s}(f)|^2. \quad (5)$$

From the definition we have

$$\hat{a}_L(f)=\int_{-\infty}^{\infty}g_L(\tau)(-1)^{a_j}\delta(\tau+T_j)e^{-i2\pi f\tau}d\tau=\sum_{j=-J_1}^{J_2}(-1)^{a_j}e^{i2\pi fT_j}, \quad (6)$$

where $-J_1$ and J_2 are the largest integers such that $|T_{-J_1}|,|T_{J_2}|<L$; i.e., $J_1+J_2=2L/T$, where $T=\langle T_j-T_{j-1}\rangle$ is the mean gap between events. From (6), we compute

$$|\hat{a}_L(w)|^2=(J_1+J_2+1)+\textstyle\sum_{j,k}(\text{cross terms}), \quad (7)$$

where typical "cross terms" have the form $(-1)^{a_j+a_k}e^{i2\pi f(T_j-T_k)}$, and, since a_j and T_j are independent random variables, by the central limit theorem the sum of these terms is $o(L)$. Thus, substituting into (5), we obtain

$$E_x(f)=(1/T)|\hat{s}(f)|^2. \quad (8)$$

Goldshtik[7] gives an alternative derivation. For the example in question, from (2) we have

$$E_x(f)=(2\pi^2/T)\mathrm{sech}^2(\pi^2 f)\sim(8\pi^2/T)\exp(-2\pi^2 f),\quad f\text{ large.} \quad (9)$$

We remark that if phase coherence exists (T_j is random but a_j is not) then peaked spectra typical of "noisy periodicity" are predicted.[7]

It remains to estimate T, the mean gap between passages around either Γ_{-1} or Γ_{+1}. Fix a neighborhood U of (0,0) of size μ (Fig. 1). Solutions leaving U through R_0 or L_0 return to U via R_i or L_i after time $\tau_0\approx 1$ which is independent of ϵ to leading order. Within U, the time spent is controlled by how close solutions are to the stable manifolds of (0,0) on entry. Linear estimates show that this time is $\lambda_+^{-1}\ln(\mu/d)$, where $d<\mu$ is the distance from the stable manifold and $\lambda_+>0$ is the expanding eigenvalue of (0,0). d is controlled by the splitting of the manifolds, which the Melnikov calculation shows to be $O(\epsilon M)$ [Eq. (3)].[2,10] Assembling this information, we find that the typical gap between structures is

$$T\approx\text{const}-\lambda_+^{-1}\ln\{\epsilon\max_\theta\|M(\theta)\|\}.$$

In our example, with $\delta=4\beta/5$, we have $M(\theta)=\sqrt{2}\gamma\pi\nu\,\mathrm{sech}(\frac{1}{2}\pi\nu)\sin\nu\theta$ and

$$T\approx\text{const}-\lambda_+^{-1}\ln\{\epsilon\gamma\nu\,\mathrm{sech}(\tfrac{1}{2}\pi\nu)\}, \quad (10)$$

where $\lambda_+\approx 1+\epsilon\delta/2$. Equations (10) and (9) provide our estimates for the power spectrum. The main points are that $E_x(f)$ decays *exponentially* with f, that this functional form is governed by the unperturbed homoclinic orbits $\Gamma_{\pm 1}$, and that the level of $E_x(f) \sim 1/T$ is relatively insensitive to all parameters *except* the (circular) frequency ν of the excitation. We remark that the decay of $E_x(f)$ implies exponential decay of correlation functions.

Numerical experiments were conducted to investigate the predictive value of the theory. Fourth-order Runge-Kutta solutions of (1) were generated with 32–64 integration steps per period $2\pi/\nu$. The resulting time history $x(t)$ was divided into 16 sequential records, each of duration 1613 seconds (4096 data points), which were fast Fourier transformed and averaged to yield power spectra. Double precision arithmetic was used throughout.

For $\gamma=0$, Melnikov theory predicts that, for parameters β and δ near which $M=0$ $(\delta=4\beta/5)$, the autonomous perturbed system has an attracting double homoclinic cycle: i.e., $\Gamma_{+1} \cup \Gamma_{-1}$ is an "infinite period" attractor.[2,10] The precise value of $\epsilon\beta$ used was found by numerical search and corresponded to the longest-period motions found. Note how close it is to prediction, despite the size of the parameters. Next, $\epsilon\gamma$ was increased to 0.001, 0.01, and 0.1 in turn and ν varied from 0.5 to 6. Typical durations between homoclinic events were computed to test Eq. (10), which was fitted by a single determination of the unknown constant at $\epsilon=0.01$, $\nu=5$, with use of $\lambda_+ \approx 1+\epsilon\delta/2 = 1.2$. Figure 2 shows that the simple theory behaves reasonably well.

Power spectra were then computed—typical examples are shown, with a time series, in Fig. 3. The general form of the power spectrum is predicted well; in particular the asymptotic slope of $\log_{10}[E_x(f)]$, $2\pi^2 \log_{10}e$ [Eq. (9)], is close to that observed. Spectral levels were obtained from the mean-gap estimate of Eq. (10) after the single fit described above. We observe that the theory consistently over (under) estimates $E_x(f)$ at low (high) frequencies. We suspect that this is due to the fact that the structure $\hat{x}(t)$ is significantly perturbed from that of Eq. (2) for $\epsilon\beta, \epsilon\delta \approx 0.5$. To test this, we integrated the Hamiltonian system $\delta=\beta=0$, with $\epsilon\gamma=0.001$, 0.01, 0.1 and obtained spectra whose asymptotic slopes lay within 2% of the prediction.[13] Integrations with lower values of $\epsilon\beta$ and $\epsilon\delta$ show a similar trend. We note that $|E_x(f)|$ covers over ten decades and that, above 1.2 Hz, spectra drop into the numerical noise floor.[13]

We then set $\beta=0$ and studied the forced, damped Duffing equation.[2,10] Here there is proof of transverse homoclinic orbits and chaotic invariant sets represent-

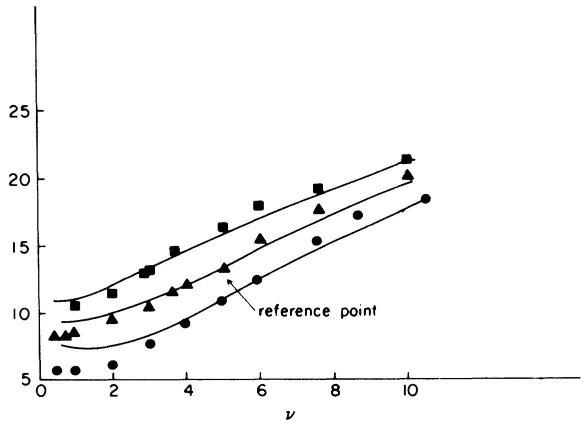

FIG. 2. Mean gaps between maxima of $|x(t)|$ as functions of force level $\epsilon\gamma$ and frequency ν: $\epsilon\gamma=0.001$ (squares), $\epsilon\gamma=0.01$ (triangles), $\epsilon\gamma=0.1$; solid lines, Eq. (10).

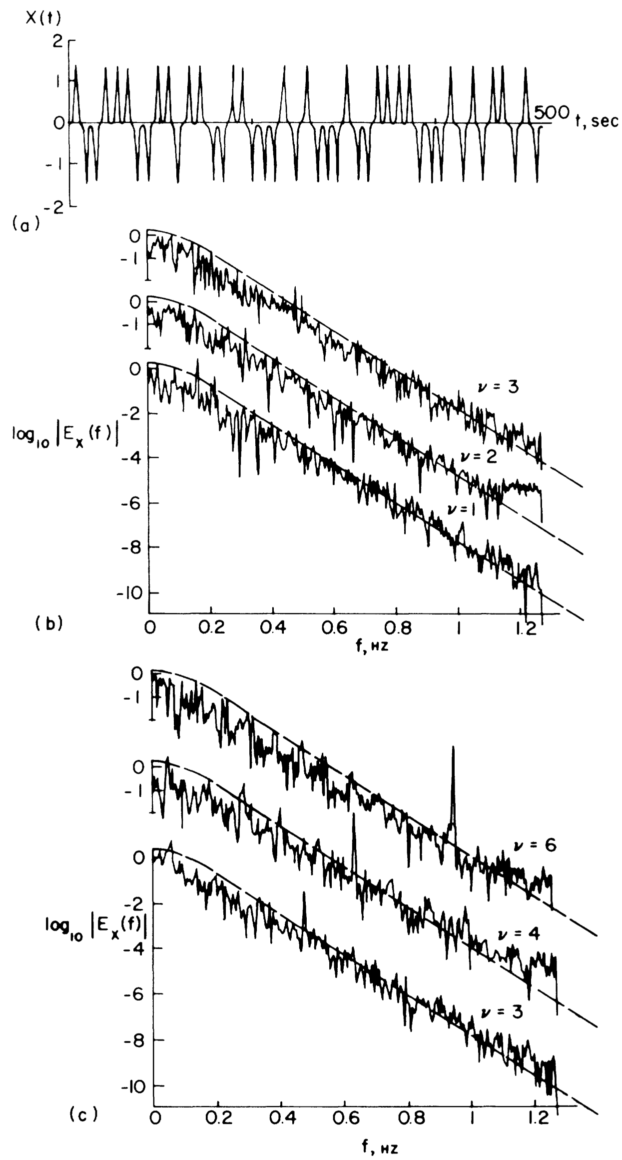

FIG. 3. (a) A time history of Eq. (1) for $\epsilon\delta=0.4$, $\epsilon\beta=0.498005$, $\epsilon\gamma=0.001$, $\nu=1$. (b) Power spectra for $\epsilon\delta=0.4$, $\epsilon\beta=0.0498005$, $\epsilon\gamma=0.01$, and various ν. (c) Power spectra for $\epsilon\gamma=0.1$ and various ν. Scales are displaced for clarity and theory of Eqs. (9) and (10) is plotted as dashed lines.

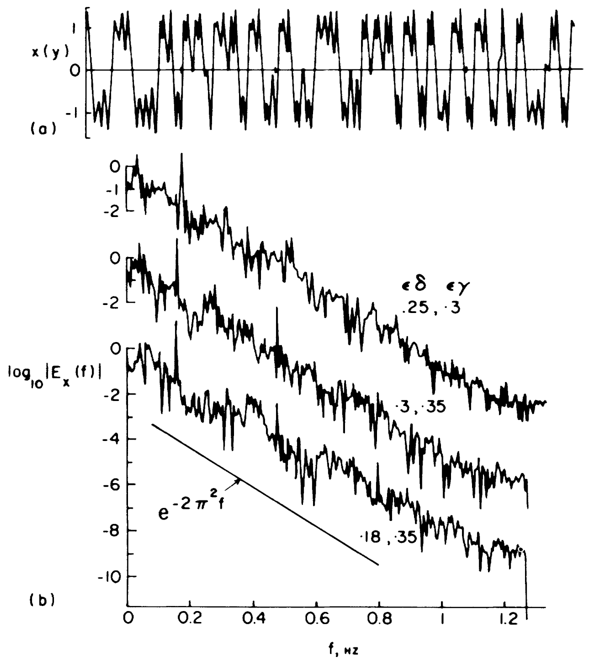

FIG. 4. (a) A time history of Eq. (1) for $\epsilon\delta=0.25$, $\epsilon\beta=0$, $\epsilon\gamma=0.3$, $\nu=1$. (b) Power spectra for $\epsilon\beta=0$, $\nu=1$, and various $\epsilon\delta, \epsilon\gamma$.

able as (4) for $\gamma > [4\delta/3\gamma\nu(2\pi)^{1/2}]\cosh(\frac{1}{2}\pi\nu)$ and ϵ sufficiently small [cf. Eq. (3)], and for $\nu \approx 0.8-1.3$ and a range of $\epsilon\gamma$ and $\epsilon\delta$ around 0.2–0.4 chaotic motions are observed. However, solutions can now stray far from the unperturbed homoclinic loops $\Gamma_{\pm 1}$, as the time series of Fig. 4(a) indicates. Nonetheless, our theory is still useful although it is now impossible to estimate T (it is unclear what a "structure" is or how the "mean gap" should be interpreted [Fig. 4(a)]. Consequently, the spectral levels cannot be predicted, but the forms and slopes agree well with Eq. (9); see Fig. 4(b).

To summarize: The assumption of randomly superposed deterministic structures leads to a simple prediction of the power spectral density of a chaotic signal. The functional form of the spectrum is given by the Fourier transform of an individual structure and its level is inversely proportional to the mean gap between the structures appearing in the signal. When the signal is the solution of a perturbed ordinary differential equation possessing homoclinic orbits to a hyperbolic saddle point, both the mean gap and the spectral form can be computed *a priori*. The simple theory is in good agreement with numerical simulations.

A detailed account of this analysis, more examples, and an extension to signals containing multiple structures will be given subsequently.[13]

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