Duality in Statistical Mechanics and String Theory

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A closed (bosonic) string propagating in a flat space-time where one of the dimensions is curled up into a circle has a Kramers-Wannier duality symmetry identical to that possessed by the Villain model. This is also similar to duality in string theory (in the sense of modular invariance). One is also led naturally to extend the space-time coordinate system by introducing dual coordinates. One of the consequences is that the radius of the internal manifold does not have a coordinate-independent meaning and one cannot physically distinguish between a radius R and a radius α'/R.

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The close analogy between the X-Y model in its Villain form and string theories in certain space-time backgrounds has been exploited recently in order to derive the equations of motion of the tachyon[1] and to derive a bound on the radius of compactification.[2] In this Letter I pursue this analogy further and study some other consequences for strings propagating in flat space-times with one of the dimensions curled up into a circle (of radius R).

First it will be shown that the order and disorder variables of the X-Y model[3] are the vertex operators that create the Kaluza-Klein (KK) and solitonic modes of the closed string, respectively. The duality that interchanges the order and disorder variables and also the high- and low-temperature phases is also the $\sigma\leftrightarrow\tau$ duality (modular invariance) in string theory that interchanges solitons and KK modes and R with α'/R. This duality symmetry, $R\leftrightarrow\alpha'/R$, of the string partition function has been noted by several authors.[4] Through use of arguments similar to those of Kadanoff and Ceva,[5] it can be shown[3] that for the X-Y model described by the action

$$\frac{R^2}{2\pi}\int d^2\sigma\,\partial_\alpha X\,\partial^\alpha X, \tag{1}$$

e^{iX} and e^{iX^*} are the order and disorder operators, respectively, where X^* is defined by

$$\partial_\alpha X=\varepsilon_{\alpha\beta}\,\partial_\beta X^*(1/2R^2). \tag{2}$$

Consider the mode expansion for a string with momentum p in the internal direction and winding number L $[X=\frac{1}{2}(X_L+X_R)]$:

$$X=x_0+p\tau+L\sigma+\tfrac{1}{2}i\textstyle\sum_n[(\alpha_n/n)e^{-2in(\tau+\sigma)}+(\tilde{\alpha}_n/n)e^{-2in(\tau-\sigma)}], \tag{3a}$$

$$X_L=x_0+x_0^*+(p+L)(\tau+\sigma)+\textstyle\sum_n(i/n)\alpha_n e^{-2in(\tau+\sigma)}, \tag{3b}$$

$$X_R=x_0-x_0^*+(p-L)(\tau-\sigma)+\textstyle\sum_n(i/n)\tilde{\alpha}_n e^{-2in(\tau-\sigma)}. \tag{3c}$$

Note that an additional zero mode x_0^* which is not present in the mode expansion for X has been introduced. This is not usually done[6,7] (except in the heterotic construction[8]) but is in fact crucial for us since we are going to make x_0^* the "position" variable conjugate to the winding number L, by imposing the commutation rules $[x_0^*,L]=-i\hbar$. Besides being an obvious and natural thing to do, this is also essential if one requires that the scattering amplitudes factorize nicely into left- and right-moving parts, each of which resembles an open-string scattering amplitude.[9] The introduction of an extra coordinate is of some physical significance. The coordinate x_0^* should not be looked upon as being less fundamental in any way than the coordinate x_0. There is a complete isomorphism between the pairs of variables (x_0^*,L) and (x_0,p). As we shall see below, one is led to extend the requirement of invariance under the usual coordinate transformations to include invariance under the discrete transformation $x_0\rightarrow x_0^*$. This will turn out to have interesting physical consequences.

The operator e^{ikx} is the vertex operator for a tachyon with momentum k in the internal direction, and $\exp[i\frac{1}{2}k(X_L-X_R)]$ that for a tachyon with winding number k. The easiest way to see that these are the right operators is to study their asymptotic behavior as $t\rightarrow-\infty$ (where $t=i\tau$ is the Wick-rotated time). Acting on the vacuum these operators reduce to e^{ikx_0} and $\exp(ikx_0^*)$, respectively. Since $[L,x_0^*]=i\hbar$, $\exp(ikx_0^*)$ changes the winding number of a state by k. Finally $\exp[\frac{1}{2}i(X_L-X_R)R^2/\alpha']$ is precisely e^{iX^*}, the disorder variable of the X-Y model. The duality $\sigma\leftrightarrow\tau$ results in an interchange of p with L and thus a KK mode with a soliton. This is obvious from Eq. (3a). A comment at this point is appropriate: It is more useful to think of this transformation as an "active" rather than a passive one in that we are not rewriting the same configuration in a different (σ,τ) coordinate system but describing a different configuration in the same coordinate system.[10] Thus a KK mode is really transformed into a soliton and vice versa. Note that after the transformation the variable x_0 no longer has an interpretation as a position coordinate since it is conjugate not to the momentum but to the winding number. Thus e^{ikx_0} represents, after the transformation, a soliton, and $\exp(ikx_0^*)$ represents a KK mode. This is of course what a duality transformation in the sense of statistical mechanics should do —interchange order and disorder variables. Since $p=M/R$ and $L=NR$, M,N integers, clearly the new configuration is allowed only if the radius is now α'/R rather than R. This is equivalent to changing the temperature T to $1/T$. That this transformation is a symmetry of the free theory is obvious from the form of the Hamiltonian:

$$H=\left[\sum_n(a_n^\dagger a_n+\tilde{a}_n^\dagger\tilde{a}_n)+\tfrac{1}{4}\left(\frac{M^2}{R^2}+\frac{R^2N^2}{\alpha'^2}\right)-2\right], \tag{4}$$

which is manifestly invariant under $R\leftrightarrow\alpha'/R$ and $M\leftrightarrow N$. Similarly the action (1) is invariant under the interchange

$$\partial_\alpha x\leftrightarrow\varepsilon_{\alpha\beta}\partial_\beta x^*(\alpha'/2R^2),$$

and $R\leftrightarrow\alpha'/R$. We now turn to the interacting theory. The tree-level n-particle amplitude is of the form[6,9]

$$\int d\mu(z_i)\prod_{\substack{i=1\\ i<j}}^{n}(z_i-z_j)^{\frac{k_{iL}k_{iL}}{4}}(z_i^*-z_j^*)^{\frac{k_{iR}k_{iR}}{4}}, \tag{5}$$

where we are interested principally in the momentum dependence. If the particles have internal momenta then $k_{iL}=k_{iR}$, and if they have winding number $k_{iL}=-k_{iR}$. In either case the result is the same because of the quadratic dependence on the momenta. Thus we conclude that at the tree level the n–KK-mode scattering amplitude (with radius R) is equal to the n-soliton scattering amplitude (with radius α'/R). Now we turn to the one-loop case. The zero-mode contribution is in the form of an integral,[6]

$$\int dp(z_1)^{p_{1L}^2/8}(z_2)^{p_{2L}^2/8}\cdots(z_n)^{p_{nL}^2/8}(z_1^*)^{p_{1R}^2/8}(z_2^*)^{p_{2R}^2/8}\cdots(z_n^*)^{p_{nR}^2/8}, \tag{6}$$

$$p_{1L,1R}=p_{L,R},\quad p_{2L,2R}=p_{L,R}-k_{1L,1R},\ \ldots,\ p_{nL,nR}=p_{L,R}-k_{1L,1R}-k_{2L,2R}-\cdots-k_{nL,nR}. \tag{7}$$

When there are compactified dimensions one has to replace dp by a discrete sum over momenta and winding numbers. Thus

$$p_L=M/R-NR,\quad k_{1L,1R}=m_1/R,\quad k_{2L,2R}=m_2/R,\ \ldots,\quad p_R=M/R+NR. \tag{8}$$

Substituting (8) into (7) and (6), one finds easily that the result has a symmetry (KK mode)$\leftrightarrow$soliton and $R\leftrightarrow\alpha'/R$.[11] Does the X-Y model have this symmetry? The answer, not surprisingly, is yes. A version of this symmetry was proved by José *et al.*[12] where the analogs of the KK mode and soliton were a symmetry-breaking perturbation and a vortex, respectively. The partition function of the X-Y model in a background of these configurations was shown to have this duality property. Having established the connection between duality in string theory and in statistical mechanics we turn to a discussion of the physical implications of this symmetry for a universe described by a closed-string theory. One has, of course, to make a distinction between the usual symmetries encountered in particle physics and the duality symmetries. In the former, one does not transform the parameters of the Lagrangean—only the dynamical degrees of freedom transform. In string theory, however, this distinction is blurred because some of the parameters of the Lagrangean become, on second quantization, dynamical degrees of freedom. In fact this is how general coordinate invariance arises from the global Poincaré symmetry of the first-quantized action of a string in a background gravitational field. Under this symmetry both the coordinates x^i *and* the metric G^{ij} transform. Thus the (radius)^2 R^2 is to be thought of as being $\langle G^{II}\rangle$ where I is the internal direction. We have already seen that at the formal level this has the consequence that the dual coordinate x_0^* is completely isomorphic to x_0 and, by the same token, one requires that the theory be invariant under the coordinate transformation $x_0\to x_0^*$. However, we have also seen that this induces also the transformation $R\to\alpha'/R$. This is rather counterintuitive since $\alpha'/R\to\infty$ as $R\to 0$.

To explore this let us ask ourselves how we would determine the radius of an internal dimension. One obvious method that we are used to is to let a particle have a momentum in the internal dimension so that it has a wave function $e^{i(M/R)x}$ and determine R by measuring its mass M/R. However, if the experimenter has no momentum in the internal dimension, x_0, he has no way of telling what the x_0 dependence of the wave function of this particle is. He can only measure its mass. He cannot therefore decide whether he should use the formula $(\text{mass})^2=M^2/R^2$ or M^2R^2/α', the latter corresponding to a particle with a winding number M in the internal dimension and wave function $\exp[i(MR/\alpha')x_0^*]$. Thus he would not be able to decide in this way whether the radius is R or α'/R. He could next try to determine whether this particle is a soliton or a KK mode by scattering other particles, say photons, but then he has to know whether this new particle is a soliton or a KK mode since, as we have seen, scattering amplitudes are completely symmetric under the interchange soliton$\leftrightarrow$KK mode and $R \leftrightarrow \alpha'/R$. Thus, at some point, he has to adopt a convention and assign to a given particle either a momentum or a winding number. This is equivalent to choosing between a wave function $\exp(ikx_0)$ and $\exp(ikx_0^*)$, i.e., *choosing a coordinate system*. In particular this means that two different experiments could adopt different coordinate systems and come up with different values of the radius (i.e., R or α'/R) and they would both be right! Thus the radius is a coordinate-dependent quantity.[13]

If the radius were large enough we could walk into the extra dimension. Could we not then distinguish between R and α'/R? The answer is no; because if we wanted to be perverse we could pretend that instead of having momentum $1/R$ with R very large, we had a winding number R with R very small. As with any symmetry there is no objectively "correct" choice. There is a complete isomorphism between the two sets of variables and it does not matter which one we choose to use to describe the phenomenon. As an analogy, if the world were one big Ising model there would be no sense in trying to decide whether you were in a high-temperature phase or a low-temperature phase because you would never be able to distinguish by any measurement (if you were part of the system) between an order variable and a disorder variable. It is not unlike any other symmetry except for the fact that it seems to relate configurations that one is used to thinking of as being physically distinct.

There are a number of questions that arise. Does this symmetry have a dynamical content when one goes over to the second-quantized formalism? If the vacuum is not invariant under this symmetry, should one expect to have domain walls separating different regions of the universe? It would also be interesting to explore the generalization of this symmetry in the group manifold compactification schemes.[14,15]

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[10]Thus $\tilde{\beta}_n \equiv \tilde{\alpha}_{-n}$ becomes the annihilation operator for the new Fock vacuum and we have a different but isomorphic set of operators and states. I thank S. Das for a discussion on this issue.

[11]In Ref. 6 the measure $1/R$ was used which explicitly broke this symmetry. However, this measure, while appropriate for open strings, is not forced upon us for closed strings. In any case this cancels the R dependence of the coupling constant.

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[13]This is of course over and above but not unlike the usual coordinate dependence of the metric associated with special relativity.

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