Statistical Mechanics of Cosmic Strings

David Mitchell and Neil Turok
Department of Theoretical Physics, The Blackett Laboratory, Imperial College, London SW7 2BZ, United Kingdom
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An analytic approach to the phase space for a network of cosmic strings is presented, based on earlier work of Frautschi and Carlitz. It correctly predicts the main features of the network at formation, and is in good agreement with the picture emerging from string simulations. Our results also have important implications for superstrings or heterotic strings in the early Universe.

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The idea that topologically stable strings formed at a phase transition in the very early Universe later seeded the formation of galaxies and clusters of galaxies has attracted much interest recently.[1] Numerical simulations of the formation[2] and evolution[3] of a network of strings have already enabled a series of direct comparisons with observational data to be made. In particular the two-point correlation function for loops chopped off a string network appears to match the observed correlation function for poor clusters, rich clusters, and superclusters remarkably closely without any free parameters.[4] If the mass per unit length μ is given by $G\mu \simeq 10^{-6}$, where G is Newton's constant, then loops with the number density of Abell clusters have the correct mass to form Abell clusters and similarly for loops with the number density of galaxies.[5]

It is important to develop an analytical approach to cosmic string formation and evolution. So far Frieman and Scherrer have discussed a random-walk model for string formation,[6] and Kibble[7] and Bennett[8] an analytic formalism for the evolution of String networks.

In this Letter we propose a new and very different approach. It predicts the main features of string formation correctly and indicates that a string-dominated universe of the kind which Kibble has discussed[7] is very unlikely, in agreement with the numerical simulations.[3]

Consider a box containing string in Minkowski space-time. In the microcanonical ensemble, we try to find which string configurations dominate the density of states. With the assumption of ergodicity, we expect the network to end up in these configurations.

Classically, a small loop has as many possible configurations as a large loop. Thus it seems obvious that one maximizes the number of states available by putting all the energy into the smallest possible loops. However, this is incorrect. In counting states one has to put a measure on phase space—effectively by quantizing the system. What this does is to set a small-scale cutoff to the size of wiggles on the string, given roughly by $\mu^{-1/2}$. Thus small loops have fewer possible states than large loops.

We shall find different behavior in two regimes. Low density corresponds to $\rho \ll \mu^2$, while high density corresponds to $\rho \simeq \mu^2$. Both regimes are of interest—cosmic strings are in the high-density regime at formation but as the Universe expands are quickly led into the low-density regime.

We ignore the energy in string-string interactions. In the low-density case the length involved in string interactions is a small fraction of the total so that this is a good approximation. The only role of the interactions is in allowing the system to explore phase space.

String statistical mechanics was considered in the context of hadronic physics by Frautschi[9] and Carlitz.[10] Recently, Bowick and Wijewardhana[11] have discussed superstrings and heterotic strings at high densities.

We want to quantize closed and, for simplicity, orientable bosonic strings in $d=4$ space-time dimensions. In the covariant quantization, the Virasoro constraints do eliminate all negative-norm states for $d \le 26$ but for $d<26$ they also create some positive-norm states which do not have classical analogs. We clearly should not count these states. In the light-cone quantization, the Lorentz algebra does not close for $d<26$, and so the S-matrix elements are not Lorentz invariant. However, since we merely want to count states, this should not be a problem.[12] We therefore only consider states created by the $D=d-2$ transverse-mode oscillators.

As is well known, for free closed bosonic strings the position of the string is given by the operator[12]

$$x^j(\sigma) = q^j + \frac{i}{2(\mu\pi)^{1/2}} \sum_{\substack{n=\infty \\ (n\neq 0)}}^{\infty} \left(\frac{\alpha_n^j e^{2in\sigma}}{n} + \frac{\tilde{\alpha}_n^j e^{-2in\sigma}}{n} \right), \tag{1}$$

where j runs from 1 to D, q^j is the center-of-mass coordinate, and $\alpha_n^j, \tilde{\alpha}_n^j$ are the mode oscillators obeying $[\alpha_n^i, \alpha_m^j] = n\delta^{ij}\delta_{m+n,0}$ and similarly for $\tilde{\alpha}_n^j$. σ runs from 0 to π. The mass operator is given by

$$m^2 = 4\pi\mu(N+\tilde{N}), \tag{2}$$

where the level-counting operator is $N = \sum_{n>0} \alpha_{-n}^i \alpha_n^i$ and similarly for $\tilde{N}$. We also have the constraint $N = \tilde{N}$ on physical states.

We impose a cutoff in mass m_0 which we take to be larger than a few times $(8\pi\mu)^{1/2}$. For low-lying states neither the Nambu approximation nor the Hardy-Ramanujan formula for the level degeneracy are good approximations but more importantly we do not expect the quantum-classical correspondence to be good.

At level n (the eigenvalue of N) the number of physical states $P(n)$ is given by the Hardy-Ramanujan formula. For closed oriented strings one simply squares $P(n)$ for open strings[13,14] to find, in D transverse dimensions and at large n,

$$P(n) \approx \tfrac{1}{2}(D/24)^{(D+1)/2} n^{-(D+3)/2} \exp[2\pi(2Dn/3)^{1/2}]. \tag{3}$$

Using (2) we find the density of states

$$\rho(m)dm = cm^{-(D+2)}\exp(bm)dm,$$

where c is a constant with dimensions $\mu^{(1+D)/2}$, i.e., inverse volume, and $b=(D\pi/3\mu)^{1/2}/2$.

First we ask what typical configurations look like at level n. Consider the operator measuring the mean squared radius of the string. By symmetry we can express this just in terms of one component $x^1(\sigma)$:

$$r^2 = \frac{(D+1)}{\pi}\int_0^\pi d\sigma :[x^1(\sigma)]^2:. \tag{4}$$

We normal order the expression and subtract the part involving the center-of-mass coordinate q^μ. We now calculate, using standard Cauchy-integral techniques[15] for $n \gg 1$, the average of r^2 at level n:

$$\frac{\sum\langle\Psi\| r^2 \|\Psi\rangle}{P(n)} \approx (D+1)\left[\frac{n}{6D}\right]^{1/2}\mu^{-1}, \tag{5}$$

where the sum is over all states at level n.

Interestingly, we find $r^2 \propto n^{1/2} \propto m$. Since the mass of a string is proportional to its invariant length l we find $r^2 \propto l$. Thus typical string configurations are Brownian walks!

Now consider a box of strings. Following Frautschi[9] and Carlitz,[10] we write the total number of microstates

$$\Omega(E) = \sum_{n=1}^{\infty}\Omega_n(E), \quad \Omega_n(E) = \frac{c^n V^n}{n!}\prod_{i=1}^{n}\int_{m_0}^{\infty} dm_i \frac{\exp(bm_i)}{m_i^{D+2}}\int d^{D+1}p_i\,\delta(\textstyle\sum E_i - E)\frac{\delta^{D+1}(\sum p_i)}{V}, \tag{6}$$

for a box of volume V containing n strings with a total energy between E and $E+dE$ with zero net momentum. One can show that the momentum integrals are dominated by the nonrelativistic region where they are Gaussian[15] and thus

$$\Omega_n \approx \frac{c^n V^n}{n!\Gamma((D+1)(n-1)/2)}K_n, \quad K_n = \prod_{i=1}^{n}\int_{m_0}^{\infty} dm_i\, m_i^{-(D+3)/2}\tilde{m}^{-(D+1)/2}\theta(E-\tilde{m})(E-\tilde{m})^{\lambda}\exp(b\tilde{m}),$$
$$\lambda = (D+1)(n-1)/2 - 1, \quad \tilde{m} = \textstyle\sum m_i. \tag{7}$$

The exponential dependence on $\tilde{m}$ maximizes the integrand for $\tilde{m} \approx E$. However, the power-law dependence $m_i^{-(D+3)/2}$ maximizes the integrand for m_i close to m_0.

If $E \gg nm_0$, at least one of the m_i must be much greater than m_0. Using a saddle-point approximation, one finds that the integral is dominated by regions where one string has a large mass $\approx E$ and the rest have small masses $\approx m_0$, and neglecting numerical constants, one obtains[15]

$$\Omega_n(E) \approx \frac{c^n V^{n-1} n[(D+1)n/2]^{1/2}\exp(bE)}{n!(m_0 b)^{n(D+1)/2}E^{(D+1)/2}[E-(n-1)m_0]^{(D+3)/2}}, \tag{8}$$

whence we see that for $E \gg nm_0$ and $n \gg 1$,

$$\Omega_n \propto (cV/n)^n (m_0 b)^{-(D+1)n/2}, \tag{9}$$

which is greatest for

$$n_{\max} \approx cV/(m_0 b)^{(D+1)/2} \tag{10}$$

—there is a fixed number density of loops independent of the total energy.

However, consistency demands that $n \ll E/m_0$, i.e.,

$$\rho = E/V \gg cm_0(m_0 b)^{-(D+1)/2}. \tag{11}$$

So the result is only valid for high densities.[9,10]

For low densities we expect Ω_n to be largest for $n \approx E/m_0$. Nevertheless we find that (8) is still valid almost right up to the maximal value $n_{\max} = E/m_0$, and so

$$\Omega \approx \Omega_{n_{\max}} \approx \left[\frac{cVm_0}{E(m_0 b)^{(D+1)/2}}\right]^{E/m_0}. \tag{12}$$

Now we turn to calculating the distribution of loop masses. The probability of finding a loop of mass between M and $M+dM$ in any given microstate is just the fraction of all accessible microstates which contain a loop with mass between M and $M+dM$, obtained by the insertion of δ functions $\sum_{i=1}^{n}\delta(m_i - M)$ into (6). Not surprisingly we find that the number of states with n loops of total energy E to $E+dE$ and one with a mass between M and $M+dM$ is approximately given by

$$\Omega_n(E,M) \approx n\Omega_1(M)\Omega_{n-1}(E-M). \tag{13}$$

We have to sum over n to obtain the number of loops with masses M to $M+dM$,

$$n(M)\propto \sum_{n=1}^{\infty} \Omega_n(E,M). \qquad (14)$$

For high densities we find (for $M \ll E$)

$$n(M)\propto n_{\max} M^{-(D+3)/2}\exp(bM)\exp[b(E-M)] \propto M^{-(D+3)/2}. \qquad (15)$$

In our discussion above we saw that $M\propto r^2$, the mean square radius, and so the number of loops of radius r to $r+dr$ is

$$n(r)dr \propto \frac{dr}{r} r^{-(D+1)}. \qquad (16)$$

This is an important result—the distribution of loop sizes is *scale invariant* [on dimensional grounds $n(r)dr$ has dimensions $r^{-(D+1)}$ since there are $D+1$ spatial dimensions and (16) indicates that no other dimensional scale enters into the expression]. We also find that the energy of the one large loop is $\approx E-m_0c \times V(m_0b)^{-(D+1)/2} \approx E$.

For low densities we find instead (for $M \ll E$ again)

$$\Omega \approx (E/m_0)\exp(bM)M^{-(D+3)/2}\exp[b(E-M)]E^{-(D+1)/2}W,$$
$$W=[cV(m_0/E)(m_0b)^{-(D+1)/2}]^{(E-M)/m_0} \equiv \exp[\alpha(E-M)/m_0], \qquad (17)$$

with $\alpha \approx \ln(m_0^4/\rho)$, and

$$n(M)dM \propto dM\, M^{-(D+3)/2}\exp(-\alpha M/m_0').$$

So large loops are exponentially suppressed.

To summarize, at high densities we find that the density of states is dominated by configurations with one large loop containing most of the energy and with a scale-invariant distribution of Brownian loops. Remarkably, these results (for $D=2$) are exactly those obtained in simulations of string formation.[2,3,6]

At first sight it is surprising that our treatment of free strings should apply to strings at formation. At this time the typical distance between strings is roughly the same as their width, so that they are certainly at high density. But the length of string involved in intersections is also a large fraction of the total length, and so neglect of the interactions is a bad approximation. However, the numerical simulations of string formation in any case do not take any account of string-interaction energy but merely assign phases at random to domains. They are also a microcanonical ensemble—for fixed string density (probability per link of forming a string) they produce typical (randomly chosen) configurations. Thus it is perhaps not too surprising (though nonetheless an important success) that we get the same result. Our result also applies for arbitrary $D>0$—as far as we know it has only ever been checked in $D=2$.

Second, at low densities we find that phase space favors configurations where all the string is chopped up into the smallest possible loops. This result is relevant to the evolution of string networks. It suggests that a box of strings in flat space-time at low densities will rapidly grind itself up into the smallest possible loops. Thus it argues strongly against the existence of an equilibrium configuration of long strings and loops where reconnection of loops balances loops being chopped off—an assumption Kibble has used to argue that the Universe could become string dominated.[16] Our results cast doubt on this scenario, and are fully consistent with the numerical simulations—phase space heavily favors chopping off of loops on scales inside the horizon where strings move as in flat space-time.

Finally, our interpretation of these results is rather different from those occurring in the literature on superstrings and heterotic strings.[17] Frautschi's result that configurations with most of the energy in a single string are favored has been interpreted as implying that the whole Universe could have come from a single string. However, we see that at high density the favored configuration for closed bosonic strings is one large string and a scale-invariant distribution of loops. Heterotic strings and closed superstrings also have a density of states given by (4), with $D=8$.[11] Thus (if we neglect interactions) all our calculations go through for these too.

In summary, we feel that the results obtained here represent a useful beginning to an analytical approach to the formation and evolution of cosmic strings.

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