Transfer-Matrix Inversion Identities for Exactly Solvable Lattice-Spin Models

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A new functional equation satisfied by the commuting row-to-row transfer matrices is derived for the eight-vertex model. Functional equations of the same form are satisfied by hard hexagons, magnetic hard squares, the self-dual Potts models, the Andrews-Baxter-Forrester models, and others. These new functional equations are called inversion identities because they generalize the inversion relation for local transfer matrices. It is conjectured that all solvable models satisfying Yang-Baxter equations possess such inversion identities and that, in general, these functional equations can be solved for the transfer-matrix eigenvalues.

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In statistical mechanics, interaction-round-a-face (IRF) models[1] are called exactly solvable if they yield a parametrized family of solutions to the star-triangle or Yang-Baxter equations. The IRF models then possess commuting families of row-to-row and corner transfer matrices whose elements are single-valued entire functions of a spectral parameter u. For these models it should be possible to obtain exact expressions for thermodynamic quantities such as free energies, correlation lengths, and interfacial tensions from the eigenvalues $V(u)$ of the row-to-row transfer matrix $\mathbf{V}(u)$. Although there are now vast hierarchies[2] of known exactly solvable models, these calculations have been successfully carried out only for a relatively few IRF models, namely, the eight-vertex model,[3] hard hexagons,[4] interacting hard squares,[5] and most recently magnetic hard squares.[6] In each case the key step was the establishment of remarkable functional equations satisfied by the row-to-row transfer matrix $\mathbf{V}(u)$, these taking quite distinct forms for the different models.

In this Letter I present functional equations of a new form called *inversion identities.* Functional equations of this form are satisfied by all of the above IRF models and others including the self-dual Potts models[7] and the Andrews-Baxter-Forrester (ABF) models.[8] The general form of an inversion identity is

$$\mathbf{V}(u)\mathbf{V}(u+\lambda)=\phi(\lambda+u)\phi(\lambda-u)\mathbf{I}+\phi(u)\mathbf{P}(u), \quad (1)$$

where u is the spectral parameter and λ is the crossing parameter. The function $\phi(u)$ is a given rational, trigonometric, or elliptic function, $\mathbf{I}$ is the identity matrix, and $\mathbf{P}(u)$ is an auxiliary matrix that commutes with $\mathbf{V}(u)$. At first sight an inversion identity appears to contain little information. However, since the elements of $\mathbf{V}(u)$ and consequently $\mathbf{P}(u)$ are all quasiperiodic entire functions, the identity places severe restrictions on the eigenvalues $V(u)$. In fact, the inversion identity almost completely determines the zeros of $V(u)$ and $P(u)$ and hence the eigenvalues themselves. I will not pursue the methods of solving these equations here; instead, I will derive the inversion identity for the prototype eight-vertex model and point out its connection with the unitarity or inversion relation[9] for local transfer matrices. In concluding, I will summarize some inversion identities for other IRF models to indicate the scope of this approach.

Let

$$s=\frac{\vartheta_1(u)}{\vartheta_1(\lambda)},\quad s_\pm=\frac{\vartheta_1(\lambda\pm u)}{\vartheta_1(\lambda)},\quad \eta=\frac{\vartheta_1^2(\lambda)}{\vartheta_4^2(\lambda)},$$
$$c=\frac{\vartheta_4(u)}{\vartheta_4(\lambda)},\quad c_\pm=\frac{\vartheta_4(\lambda\pm u)}{\vartheta_4(\lambda)},\quad \mu=\frac{\vartheta_4(0)}{\vartheta_4(\lambda)}, \quad (2a)$$

where $\vartheta_1(u)$ and $\vartheta_4(u)$ are standard elliptic theta functions[10] of nome $q=e^{\pi i\tau}$. Then the four independent face weights of the eight-vertex model can be written as

$$\begin{aligned}
\omega_1&=W(\sigma_1,\sigma_2,\sigma_1,\sigma_2)=\mu^{-1}cc_-,\\
\omega_2&=W(\sigma_1,\sigma_2,-\sigma_1,-\sigma_2)=\eta\mu^{-1}ss_-,\\
\omega_3&=W(\sigma_1,\sigma_2,\sigma_1,-\sigma_2)=\mu^{-1}cs_-,\\
\omega_4&=W(\sigma_1,\sigma_2,-\sigma_1,\sigma_2)=\mu^{-1}c_-s,
\end{aligned} \quad (2b)$$

where $\sigma_1,\sigma_2=\pm 1$ and the face spins are in anticlockwise order starting at the bottom left-hand corner. In the ferromagnetic regime u, λ, and τ are all pure imaginary with $0<q=e^{\pi i\tau}<1$ and $0<\mathrm{Im}u<\mathrm{Im}\lambda<(\pi/2)\mathrm{Im}\tau$. The elements of the row-to-row transfer matrix are

$$V(\sigma|\sigma')=\prod_{j=1}^{N} W(\sigma_j,\sigma_{j+1},\sigma'_{j+1},\sigma'_j), \quad (3)$$

where σ and σ' are the configurations of two successive periodic rows of N spins. The elements of $\mathbf{V}(u)$ and its eigenvalues are entire functions of the complex variable u with quasiperiodicity determined by (2). The transpose matrix $\mathbf{V}^{\mathrm{T}}(u)=\mathbf{V}(\lambda-u)$ corresponds to rotation of the lattice through 90° and, when $u=0$, $\mathbf{V}(u)$ and $\mathbf{V}(\lambda-u)$ reduce to shift operators.

Let $\mathbf{V}=\mathbf{V}(u)$, $\mathbf{V}'=\mathbf{V}(u+\lambda)$ and similarly for the face weights W and W'. Then it follows that

$$[\mathbf{VV}'](\sigma|\sigma') = \mathrm{Tr}\mathbf{S}(\sigma_1,\sigma_2,\sigma_2',\sigma_1')\mathbf{S}(\sigma_2,\sigma_3,\sigma_3',\sigma_2')\cdots\mathbf{S}(\sigma_N,\sigma_1,\sigma_1',\sigma_N'), \tag{4a}$$

where the sixteen 2×2 $\mathbf{S}$ matrices have elements

$$[\mathbf{S}(\sigma_1,\sigma_2,\sigma_2',\sigma_1')](\tau_1,\tau_2) = W(\sigma_1,\sigma_2,\tau_2,\tau_1)W'(\tau_1,\tau_2,\sigma_2',\sigma_1'). \tag{4b}$$

Explicitly, the $\mathbf{S}$ matrices are given by

$$\mathbf{S}(\sigma_1,\sigma_2,\sigma_2,\sigma_1) = \mathbf{X}(\sigma_2)\mathbf{A}\mathbf{X}(\sigma_1),\quad \mathbf{S}(\sigma_1,\sigma_2,-\sigma_2,-\sigma_1) = \mathbf{X}(\sigma_2)\mathbf{B}\mathbf{X}(\sigma_1),$$
$$\mathbf{S}(\sigma_1,\sigma_2,\sigma_2,-\sigma_1) = \mathbf{X}(\sigma_2)\mathbf{C}\mathbf{X}(\sigma_1),\quad \mathbf{S}(\sigma_1,\sigma_2,-\sigma_2,\sigma_1) = \mathbf{X}(\sigma_2)\mathbf{D}\mathbf{X}(\sigma_1), \tag{5a}$$

where $\sigma_1,\sigma_2=\pm 1$,

$$\mathbf{A} = \frac{1}{\mu^2}\begin{bmatrix} c^2c_+c_- & -c_+c_-s^2 \\ c^2s_+s_- & -\eta^2s^2s_+s_- \end{bmatrix},\quad \mathbf{B} = \frac{cs}{\mu^2}\begin{bmatrix} -\eta c_-s_+ & c_-s_+ \\ -c_+s_- & \eta c_+s_- \end{bmatrix},$$
$$\mathbf{C} = \frac{cs}{\mu^2}\begin{bmatrix} -c_+c_- & c_+c_- \\ -\eta s_+s_- & \eta s_+s_- \end{bmatrix},\quad \mathbf{D} = \frac{1}{\mu^2}\begin{bmatrix} c^2s_-s_+ & -\eta c_-s^2s_+ \\ c^2c_+s_- & -\eta c_+s^2s_- \end{bmatrix}, \tag{5b}$$

and

$$\mathbf{X}(1) = \mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\quad \mathbf{X}(-1) = \mathbf{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{5c}$$

The orthonormal eigenvectors of the local spin-flip operator $\mathbf{X}$ are

$$\mathbf{x}_+ = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix},\quad \mathbf{x}_- = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix},\quad \mathbf{x}_-^{\mathrm{T}}\mathbf{x}_+ = 0. \tag{6a}$$

These vectors are respectively right and left eigenvectors of $\mathbf{A}$ and satisfy

$$\mathbf{X}\mathbf{x}_+ = \mathbf{x}_+,\quad \mathbf{A}\mathbf{x}_+ = (c_+c_-s_+s_-)\mathbf{x}_+,\quad \mathbf{x}_-^{\mathrm{T}}\mathbf{X} = -\mathbf{x}_-^{\mathrm{T}},\quad \mathbf{x}_-^{\mathrm{T}}\mathbf{A} = \mu^{-4}(1-\eta^2)c^2s^2\mathbf{x}_-^{\mathrm{T}}. \tag{6b}$$

It follows that the diagonal elements of (4a) are given by

$$[\mathbf{VV}'](\sigma|\sigma') = (c_+c_-s_+s_-)^N + [\mu^{-4}(1-\eta^2)c^2s^2]^N. \tag{6c}$$

For the off-diagonal elements, observe that

$$\mathbf{CX} = -\mathbf{C},\quad \mathbf{CA} = \mu^{-4}(1-\eta^2)c^2s^2\mathbf{C}, \tag{6d}$$

and

$$\mathbf{CD} = \frac{\Delta c^2s^2}{\mu^6}\begin{bmatrix} -c^2c_+c_- & c_+c_-s^2 \\ -\eta c^2s_+s_- & \eta s_+s_-s^2 \end{bmatrix},\quad \Delta = \frac{\mu^3\vartheta_1(2\lambda)}{\vartheta(\lambda)}. \tag{6e}$$

So, by progressively moving any $\mathbf{C}$ matrices to the right, annihilating all of the $\mathbf{A}$ matrices, and contracting the $\mathbf{C}$ matrices with the paired $\mathbf{D}$ matrices, we see that each off-diagonal element in (4a) is of the form $(cs)^N$ multiplied by an entire function of u.

The above arguments establish the inversion identity (1) for the eight-vertex model. The auxiliary matrix $\mathbf{P}(u)$ commutes with $\mathbf{V}(u)$, the elements of $\mathbf{P}(u)$ are entire functions of u, and

$$\phi(u) = (cs)^N = [\vartheta_1(u)\vartheta_4(u)/\vartheta_1(\lambda)\vartheta_4(\lambda)]^N. \tag{7}$$

Moreover, $\phi(0)=0$ and $\phi(\lambda)=1$ so that (1) is trivially satisfied when $u=0$ and $\mathbf{V}(u)$ is the shift operator. By way of comparison, Baxter's functional equation[3] for the eight-vertex model, in the present notation, becomes

$$\mathbf{V}(u)\mathbf{Q}(u) = \phi(u)\mathbf{Q}(u-\lambda) + \phi(u-\lambda)\mathbf{Q}(u+\lambda), \tag{8}$$

where $\mathbf{Q}(u)$, like $\mathbf{P}(u)$, is an auxiliary matrix commuting with $\mathbf{V}(u)$. This equation is entirely consistent with (1) and, in fact, provides a direct relation between the matrices $\mathbf{P}(u)$ and $\mathbf{Q}(u)$.

Since the common eigenvectors of $\mathbf{V}(u)$ and $\mathbf{P}(u)$ are independent of u, the functional equation (1) is satisfied by the individual eigenvalues $V(u)$ and $P(u)$. Moreover, in the ferromagnetic regime and for large N, the term $\phi(u)P(u)$ is exponentially small compared with $V(u)V(u+\lambda)$ and so the inversion identity reduces to

$$V(u)V(u+\lambda) = \phi(\lambda+u)\phi(\lambda-u), \tag{9}$$

which is the inversion relation.[11] This functional equation is satisfied, in the thermodynamic limit, by all of the eigenvalues $V(u)$, not just the principal eigenvalue. In this sense the matrix $\mathbf{V}(u+\lambda)=\mathbf{V}^{\mathrm{T}}(-u)$ is almost the inverse of $\mathbf{V}(u)$. The extra term $\phi(u)\mathbf{P}(u)$ in (1) arises from the periodic boundary condition and so, physically, we would expect it to be exponentially small at least away from criticality. Indeed, if the boundary condition in (4a) is changed to $\sigma_{N+1}=\sigma'_{N+1}=1$ and the trace is replaced by a sum over $\tau_{N+1}=\pm 1$, then one obtains precisely (1) without the term $\phi(u)\mathbf{P}(u)$. This is an immediate consequence of the inversion relation for local transfer matrices

$$\sum_{\tau_2} W(\sigma_1,\sigma_2,\tau_2,\tau_1\,|\,u)W(\tau_1,\tau_2,\sigma_2,\sigma'_1\,|\,u+\lambda) = c_+c_-s_+s_-\delta(\sigma_1,\sigma'_1), \qquad (10)$$

which explains the connection between the local inversion relation (10) and the transfer-matrix inversion identity (1).

Inversion identities obviously hold for the Ising and six-vertex models, since they are special cases of the general eight-vertex model. In the six-vertex case the elliptic functions reduce to trigonometric functions and, for the Ising case, $\mathbf{P}(u)=\mathbf{R}$ is just the spin-reversal operator.[1] Inversion identities, however, appear to hold universally for solvable IRF models. For hard hexagons[4] and interacting hard squares,[5] for example, (1) holds with $\lambda=\pi/5$, $\phi(u)=[\vartheta_1(u)/\vartheta_1(\lambda)]^N$, and $\mathbf{P}(u)=\mathbf{V}(u-2\lambda)$. Similarly, inversion identities hold for magnetic hard squares[6] on both the E and T solution manifolds. On the E manifold $\lambda=\pi/3$, $\phi(u)=[\vartheta_1(u)/\vartheta_1(\lambda)]^N$, and

$$\mathbf{P}(u)=\mathbf{V}(u+2\lambda)^{-1}\{\phi(\lambda+u)\phi(\lambda-u)[\mathbf{I}+(-1)^N\mathbf{R}]+(-1)^N\phi(\lambda-u)\mathbf{V}(u)+\phi(\lambda+u)\mathbf{R}\mathbf{V}(u+\lambda)\}, \qquad (11)$$

whereas on the T manifold, λ is arbitrary and $\phi(u)=(\sin u/\sin\lambda)^N$, precisely as for the six-vertex model. Finally, for completeness, it should be noted that inversion identities have also been obtained for the self-dual, q-state Potts models[7] and the ABF models.[8] For the Potts models it is found that $\phi(u)=(\sin u/\sin\lambda)^{2N}$ with $2\cos\lambda=\sqrt{q}$, whereas for the n-state ABF models in the lattice-gas representation, it is found that $\phi(u)=[\vartheta_1(u)/\vartheta_1(\lambda)]^N$ with $\lambda=\pi/(2n+1)$.

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