

Perturbative Stability of Smooth Strings

Robert D. Pisarski

Fermi National Accelerator Laboratory, Batavia, Illinois 60510

(Received 20 January 1987)

Polyakov's theory of surfaces embedded in Euclidean space-time with an extrinsic curvature term is stable under small fluctuations only if the number of dimensions is positive and less than 26.

PACS numbers: 11.17.+y, 68.10.Cr

Polyakov has proposed a theory of surfaces in which the action depends not only upon the area of the surface, but also on the way in which the surface is embedded in space-time, through its extrinsic curvature.¹⁻⁵ This theory serves as a model for flux sheets in QCD, particularly if it is possible to reach a phase in which surfaces are not "creased" over large distances, as they are in the Nambu model, but "smooth." This theory had arisen previously in the study of interfaces by Helfrich⁶ and others.^{7,8}

The Nambu string can be written in a way which exhibits a local conformal symmetry; in the quantum theory, this symmetry is only manifest in 26 dimensions.⁹ Smooth surfaces do not appear to possess any such conformal symmetry, so presumably they are con-

sistent theories in a wide range of dimensions. In this work I compute in which dimensions smooth surfaces are stable under small fluctuations.

A better understanding of the role played by the Liouville theory in this model is gained along the way. Förster,⁸ Polyakov,¹ and David² suggested that the Liouville action arises in the quantization of smooth surfaces. The Liouville term is proportional to $26-d$, which led David² to conjecture that smooth surfaces are unstable in more than 26 dimensions. I show this is true, but find that the Liouville term does not show up where expected. Instead of being most important over large distances,^{1,2,8} it dominates only at short distances.

In conformal gauge, where the metric $g_{ab} = \rho\delta_{ab}$, the action for smooth surfaces is

$$S = \frac{1}{2\alpha} \int d^2z \rho \left[\left(\frac{-1}{\rho} \partial^2 x \right)^2 + i \frac{\lambda^{ab}}{\rho} (\partial_a x \partial_b x - \rho \delta_{ab}) \right] + \int d^2z \mu \rho + S_{\text{ghost}}(\rho). \quad (1)$$

The surface is described by $x = x(z^a)$, where x is a vector in d Euclidean dimensions; the z^a , $a=1,2$, are the coordinates of the world sheet.

The first term in Eq. (1) is the square of the extrinsic curvature for the surface, with α a dimensionless coupling constant. In the second term, λ^{ab} is a constraint field which fixes the metric to be that intrinsic to the surface. The third is the Nambu term, with a string tension equal to μ . Lastly, the contribution of the Fadeev-Popov ghosts for general coordinate invariance is represented by $S_{\text{ghost}}(\rho)$.

Suppose that one sets $\rho=1$ in Eq. (1) and drops the integration over ρ in the functional integral. This truncated theory is a sum not over all surfaces, but only flat ones. (By flat, I mean that the intrinsic curvature vanishes. As shown by the example of a cylinder, there are many surfaces that are curved extrinsically but not intrinsically.) I have studied this model of flat surfaces before,¹⁰ and these calculations are of help here.

The x 's appear quadratically in the action, so they can be integrated out to give

$$S_{\text{eff}}(\rho, \lambda^{ab}) = \frac{d}{2} \text{tr} \ln \left[\frac{-1}{\rho} \partial^2 \left(\frac{-1}{\rho} \partial^2 \right) - \frac{i}{\rho} \partial_a \lambda^{ab} \partial_b \right] + \int d^2z \left[\mu \rho - \frac{i}{2\alpha} \rho \lambda^{ab} \delta_{ab} \right] + S_{\text{ghost}}(\rho). \quad (2)$$

The integration is with respect to the invariant measure on the world sheet, $\int d^2z \rho$.

I expand about the simplest possible background—an infinite, flat surface. Thus I assume that $\rho = \rho_0$ and $\lambda^{ab} = -i\lambda_0 \delta^{ab}$, where ρ_0 and λ_0 are constants. To determine these constants it is necessary to calculate S_{eff} under this *Ansatz*. The result is

$$S_{\text{eff}} = \rho_0 A \left[\frac{-1}{\alpha_r} \lambda_0 + \mu + \frac{d}{8\pi} \lambda_0 [1 - \ln(\lambda_0)] \right], \quad (3)$$

where A is the area of the surface. The renormalized

coupling is α_r , $\alpha_r^{-1} = \alpha^{-1} - (d/4\pi) \ln(\Lambda) + \dots$, where Λ is a momentum cutoff. This relationship between α_r and α implies that the theory is asymptotically free.^{1-4,7,8}

For the action to be stationary under variations of ρ_0 and λ_0 , $\lambda_0 = -8\pi\mu/d = \exp(-8\pi/\alpha_r d)$; the bare string tension μ must be negative so that $\lambda_0 > 0$. The value of ρ_0 is not fixed, and can be taken as an arbitrary positive number. At the stationary point, $S_{\text{eff}} = 0$, so the renormalized string tension vanishes.

These results are similar to those found in the study of

smooth surfaces in a large number of dimensions.^{2,3} At large d , the corrections to S_{eff} are of order $1/d$ and so small. I work in an arbitrary number of dimensions, where this represents merely one of many possible solutions to the equations of motion.

What, then, is the value of analyzing stability about this special solution? Because the theory is asymptotically free, perturbation theory is valid at large momenta. But large momenta corresponds to small distances, and at short distances any (regular) surface is essentially flat. Thus I establish a necessary condition for stability.

Define

$$\begin{aligned} \rho &= \rho_0 [1 + (8\pi/|d|)^{1/2}(\tilde{\rho}/m)], \\ \lambda^{ab} &= (m/\rho_0)[-im\delta^{ab} + (8\pi/|d|)^{1/2}\tilde{\lambda}^{ab}], \end{aligned} \tag{4}$$

where $m^2 = \rho_0 \lambda_0 > 0$. Note that while the stationary point of λ^{ab} is imaginary, the fluctuations in $\tilde{\lambda}^{ab}$ are real.¹¹ This is required so that integration over the $\tilde{\lambda}^{ab}$ generates the proper δ -function constraint in the functional integral.

Expanding to quadratic order in $\tilde{\rho}$ and $\tilde{\lambda}$,

$$S_{\text{eff}}(\rho, \lambda^{ab}) = \frac{1}{2} \int \int [\tilde{\rho} \Delta^{-1}(\rho, \rho) \tilde{\rho} + 2\tilde{\rho} \Delta^{-1}(\rho, \lambda) \tilde{\lambda} + \tilde{\lambda} \Delta^{-1}(\lambda, \lambda) \tilde{\lambda}] + \dots, \tag{5}$$

$\Delta^{-1}(\rho, \lambda) = \Delta^{-1}(\lambda, \rho)$. The indices on λ^{ab} are often dropped.

I start with the diagonal term for the metric field, $\Delta^{-1}(\rho, \rho)$. For this it is possible to ignore $\tilde{\lambda}$ and write

$$S_{\text{ghost}}(\rho) + \frac{d}{2} \text{tr} \ln \left[\frac{-1}{\rho} \partial^2 \right] + \frac{d}{2} \text{tr} \ln \left[\frac{-1}{\rho} \partial^2 + \frac{m^2}{\rho_0} \right] = \frac{1}{2} \int \int \tilde{\rho} \Delta^{-1}(\rho, \rho) \tilde{\rho} + \dots \tag{6}$$

The first two terms on the left-hand side are identical to those in Polyakov's formulation of the Nambu model. Their only dependence on ρ is through the conformal anomaly, so they give the usual factor of the Liouville action.⁹ The last is an integration over a massive mode.

Let the momentum be p^a : It is convenient to introduce a dimensionless variable P , which is proportional to the magnitude of the momentum, $P = p^2/m^2$, and the unit vector \hat{p}^a along p^a . Then,

$$\Delta^{-1}(\rho, \rho) \sim \text{sgn}(d) \left[\frac{26-d}{6d} P - 2 \frac{\ln(P)}{P} + \dots \right]. \tag{7}$$

The first term on the right-hand side is the Liouville action to $\sim O(\tilde{\rho}^2)$, written in an unfamiliar way. The second is the leading contribution of the massive mode at large momentum. In Eq. (7) and henceforth, I write only the dominant terms at large P ; corrections are at most $\sim 1/\ln(P)$ times the terms written.

Förster,⁸ Polyakov,¹ and David² proposed that the Liouville term appears in the quantum theory of smooth surfaces. They argued that since the Nambu term has fewer derivatives than the extrinsic curvature term, and as the Liouville theory arises from the quantization of the Nambu term, that the Liouville term should be most

significant about zero momentum.

In contrast, Eq. (7) shows that the Liouville term dominates $\Delta^{-1}(\rho, \rho)$ at large momentum. Because the contribution of the massive mode is so much smaller than that of the Liouville term at large P , presumably when $d \neq 26$ the Liouville term dominates $\Delta^{-1}(\rho, \rho)$ at large P to all orders in perturbation theory. Away from large P , however, the massive mode also contributes, and there is no reason why it cannot be as important as the Liouville term.

For example, consider the theory about $d = \infty$. (The model is unstable in this limit, but it serves to make a point.) For large d , Eq. (6) can be used to compute $\Delta^{-1}(\rho, \rho)$ around zero momentum.² Though the Liouville term contributes $-\frac{1}{6}P$, as $P \rightarrow 0$ the massive mode gives $-1 + \frac{1}{6}P + \dots$, so in all $\Delta^{-1}(\rho, \rho) \sim -1 + O(P^2)$: The Liouville term does not dominate at small momentum. This result, which supersedes that of Ref. 2, is discussed at the end of the paper.

The other terms in Δ^{-1} can be found in a straightforward manner by expanding perturbatively in $\tilde{\rho}$ and $\tilde{\lambda}^{ab}$. At large P the off-diagonal elements are

$$\Delta^{-1}(\rho, \lambda^{ab}) \sim -i \text{sgn}(d) [(\delta^{ab}/2) \ln(P) + \hat{p}^a \hat{p}^b]. \tag{8}$$

$\Delta^{-1}(\lambda, \lambda)$ can be read off directly from the results for flat surfaces,

$$\Delta^{-1}(\lambda, \lambda) \sim \text{sgn}(d) [(K^1 + K^2)/4P + \ln(P)K^3/4P - K^4/2P + 2K^5/P]. \tag{9}$$

The K 's are matrices which span the space of two symmetric tensors, and are defined in Eq. (4.8) of Ref. 10. The momentum dependence of $\Delta^{-1}(\lambda, \lambda)$ follows from Eq. (4.15) of Ref. 10.

The theory is perturbatively stable if every eigenvalue of Δ^{-1} has a real part that is positive. These four eigenvalues can be determined with a little effort.

One eigenvalue of Δ^{-1} is $\sim \text{sgn}(d) \ln(P)/2P$. For this eigenvalue to be positive so must the number of dimensions. Hence I assume that $d > 0$.

The characteristic equation for the remaining three eigenvalues is

$$y^3 - [cP - \ln(P)/P]y^2 + \frac{1}{2} \ln^2(P)y - \ln^3(P)/4P = 0, \quad (10)$$

where y represents the eigenvalue, and the constant $c = (26 - d)/6d$. In Eq. (10) I have dropped any terms which are small at large P . The only exception is that for y^2 , since when $d = 26$, $c = 0$, and the term $\sim \ln(P)/P$ is dominant.

When the number of dimensions is not equal to 26, the solutions to Eq. (10) are given by

$$y \sim cP, \quad y \sim \ln^2(P)/2cP, \quad y \sim \ln(P)/2P. \quad (11)$$

When d is less than 26 ($c > 0$), all eigenvalues are real and positive and the theory is stable. The theory is unstable when d is greater than 26 ($c < 0$), with two positive and two negative eigenvalues.

26 dimensions is a special case. There is one positive eigenvalue, and a pair of complex conjugate eigenvalues,

$$y \sim \frac{1}{2} P \ln(P), \quad (12)$$

$$y \sim \pm (i/\sqrt{2}) \ln(P) - (3/4P) \ln(P).$$

There are corrections to the complex conjugate eigenvalues that are of order 1, etc., but these are all purely imaginary. The leading real part of these eigenvalues is given in Eq. (12), and as it is negative the theory is not stable in 26 dimensions.

There is an easy way to understand these results. At least over short distances, smooth surfaces can be viewed as an amalgam of the Liouville theory (which results from integrating over the metric field) and the model of flat surfaces (as that contains the dynamics of the constraint field). The Liouville theory which arises is proportional to $26 - d$, so smooth surfaces are unstable in more than 26 dimensions.² On the other hand, flat surfaces are stable only if the number of dimensions is positive,¹² so this should hold for smooth surfaces as well. In 26 dimensions the Liouville theory vanishes, and detailed calculation is needed to show instability.

That smooth surfaces are only stable for $0 < d < 26$ is in one sense rather remarkable. For smooth surfaces, x is a vector, and so it is natural to expect that it should be possible to develop some sort of consistent solution at large d .^{2,3} This would allow the theory to be studied not just at large but over all momenta. For the Liouville theory of the Nambu string, the correct large- d limit is about *minus* infinity.¹³ For flat surfaces, the correct large- d limit is about *plus* infinity.¹² If Nature were kind, she would allow smooth surfaces to have a stable solution about either plus or minus infinity; instead, neither is allowed.

The nature of the unstable solution about infinite d can be studied by the rotation of the contours of integration for $\tilde{\rho}$ and $\tilde{\lambda}$. As mentioned following Eq. (7), in this limit $\Delta^{-1}(\rho, \rho)$ does not vanish about zero momentum. Further calculation shows that this is (usually) true for

all components of the inverse propagator Δ^{-1} , as well as for those of the propagator Δ computed from Δ^{-1} .

This is unlike the Nambu model. There, the interactions of the metric field ρ are given by the Liouville theory.⁹ In perturbation theory, which can be used for $d = -\infty$, the two-point function of ρ has logarithmic correlations over large distances.¹³

Let me make the dangerous assumption that qualitative properties of smooth surfaces for $0 < d < 26$ can be gleaned from the solution at infinite d . It is certainly natural to expect that correlations between the λ^{ab} 's are exponentially damped over large distances.^{1-3,7,8} The solution at $d = \infty$ indicates that for smooth surfaces, interactions due to the extrinsic curvature term mix up the ρ and λ^{ab} fields together, so that in the end, neither have long-ranged correlations.

Not too much should be made of these differences. What is most important is the two-point function of the x 's, since this measures the mean square size of the surface. For smooth surfaces about $d = +\infty$, generally this size increases logarithmically with the area of the surface, as it does for Nambu strings around $d = -\infty$.¹³

Nevertheless, that the model is more complicated in the infrared than first thought might indicate that Polyakov's goal—of reaching a phase in which surfaces are smooth instead of creased over large distances¹—could be easier than first thought.

The solution about $d = +\infty$ will be presented in a separate publication, along with details of the present study.

I thank F. David, and especially C. Zachos, for helpful discussions. Fermilab is operated by Universities Research Association under contract with the U.S. Department of Energy.

¹A. M. Polyakov, Nucl. Phys. **B268**, 406 (1986).

²F. David, Europhys. Lett. **2**, 577 (1986); F. David and E. Guitter, Saclay Report No. SPhP/87/89, 1987 (to be published).

³F. Alonso and D. Espriu, Nucl. Phys. **B** (to be published); E. Braaten, R. D. Pisarski, and S.-M. Tse, Phys. Rev. Lett. **58**, 93 (1987); E. Braaten and S.-M. Tse, Argonne Report No. ANL-HEP-PR-86, 1987 (to be published); P. Olesen and S.-K. Yang, Nucl. Phys. **B283**, 73 (1987).

⁴F. Alonso, D. Espriu, and J. I. Latorre, Harvard University Report No. HUTP-86/A086 (to be published); A. R. Kavalov, I. K. Kostov, and A. G. Sedrakyan, Phys. Lett. **175B**, 331 (1986); A. R. Kavalov and A. G. Sedrakyan, Phys. Lett. **182B**, 33 (1986); P. O. Mazur and V. P. Nair, Institute of Theoretical Physics, University of California, Santa Barbara, Report No. NSP-ITP-86-91 (to be published).

⁵E. Braaten and C. K. Zachos, Phys. Rev. D **35**, 1512

(1987); T. L. Curtright, G. I. Ghandour, C. B. Thorn, and C. K. Zachos, *Phys. Rev. Lett.* **57**, 799 (1986); T. L. Curtright, G. I. Ghandour, and C. K. Zachos, *Phys. Rev. D* **34**, 3811 (1986).

⁶W. Helfrich, *Z. Naturforsch. C* **28**, 693 (1973), and *J. Phys. (Paris)* **46**, 1263 (1985), and **47**, 321 (1986), and Berlin Report No. FUB-TKM 21/86, 1986 (to be published).

⁷L. Peliti and S. Leibler, *Phys. Rev. Lett.* **54**, 1690 (1985).

⁸D. Förster, *Phys. Lett.* **114A**, 115 (1986); and Berlin Report No. FUB-TKM 24/86, 1986 (to be published); H. Kleinert, *Phys. Lett.* **174B**, 335 (1986).

⁹A. M. Polyakov, *Phys. Lett.* **103B**, 207 (1981).

¹⁰R. D. Pisarski, *Phys. Rev. D* **28**, 2547 (1983).

¹¹Since the absolute value of d enters in Eq. (4), this also holds for negative values of d . Of course, the number of transverse degrees of freedom of a surface is $d - 2$, so that the mod-

el is only physical for $d > 2$. The model is defined formally when $d < 2$ by the Lagrangean of Eq. (2). While $d < 2$ is unphysical, it is comforting to find that the lower bound for stability in d , $d = 0^+$, is a safe distance from the values of interest, $d = 3$ and 4.

¹²Like smooth surfaces, the model of flat surfaces is asymptotically free. Following the methods of this work, flat surfaces can be shown to be stable at large momenta for any $d > 0$. With use of the solution of Ref. 10, they can also be shown to be stable over all momenta about $d = +\infty$.

¹³O. Alvarez, *Nucl. Phys.* **B216**, 125 (1983); B. Durhuus, H. B. Nielsen, P. Olesen, and J. L. Petersen, *Nucl. Phys.* **B196**, 498 (1982); I. K. Kostov and A. Krzywicki, LPTHE Orsay Report No. 86/48, 1986 (to be published); J. Jurkiewicz and A. Krzywicki, *Phys. Lett.* **148B**, 148 (1984); A. B. Zamolodchikov, *Phys. Lett.* **117B**, 87 (1982).