Properties of Random Superpositions of Plane Waves

P. O'Connor
Department of Physics, University of Washington, Seattle, Washington 98195

J. Gehlen
Department of Chemistry, University of Washington, Seattle, Washington 98195

and

E. J. Heller
Departments of Chemistry and Physics, University of Washington, Seattle, Washington 98195
(Received 21 November 1986)

Random superpositions of plane waves, designed to mimic the local behavior of completely random eigenfunctions of classically chaotic Hamiltonian systems, are shown to have surprising properties, including structures which may be precursors of periodic orbit scar localization.

PACS numbers: 03.65.Ge, 03.65.Sq, 05.45.+b, 42.10.−s

The nature of the quantum eigenstates of classically chaotic Hamiltonian systems is the subject of much current research.[1-3] Some of the reasons for the keen interest are (1) a desire to understand the quantum manifestations of the classical nonlinear dynamics and chaos which has been the object of so much work and progress in the last decade, (2) a desire to extend the well developed semiclassical methods of quasiperiodic dynamics into the chaotic domain, and (3) a need to extend the considerable body of results on the quantum eigenvalue spectra of classically chaotic systems[4] to the eigenfunctions, which contain more information than the eigenvalues.

For quasiperiodic classical dynamics, the semiclassical eigenfunctions can be written

$$\Psi_E(\mathbf{x}) = \sum_n a_{n,E}(\mathbf{x}) \exp[iS_n(\mathbf{x})/\hbar + i\phi_n], \qquad (1)$$

where $S_n(\mathbf{x})$ is the classical action, ϕ_n is a phase correction which depends on the caustics that the orbit touches (the Maslov phase), and $a_{n,E}(\mathbf{x})$ is a real amplitude. The quasiperiodic classical trajectory may arrive at the position $\mathbf{x}$ in several distinct ways, distinguished by distinct momenta $\mathbf{p}_n = \nabla S_n(\mathbf{x})$ (but $p = |\mathbf{p}_n| = \hbar k$ is fixed by the local kinetic energy and is independent of n) and distinct classical probability $|a_{n,E}(\mathbf{x})|^2 = \det[\partial \mathbf{p}_n / \partial \mathbf{J}]$, where $\mathbf{J}$ are the classical action variables. Equation (1) is a sum over these distinct ways of reaching $\mathbf{x}$, and is simply the familiar statement, common to so many wave equations, that in the short-wavelength limit the "rays" are the gradients of the associated wave fronts. In the short-wavelength (high energy, small $\hbar$) limit, we may locally take the waves to be plane waves with a wavelength determined (in the Schrödinger case) by the local kinetic energy, or (in the optics case) by the local refractive index.

Quasiperiodic motion implies that the sum over n in Eq. (1) contains a finite number of terms; i.e., a trajectory or ray accesses a given point $\mathbf{x}$ heading in finitely many directions. A familiar case from classical mechanics is the Lissajous motion associated with two separable oscillators of incommensurate frequencies.

Chaotic ray motion implies that each point $\mathbf{x}$ is accessed by infinitely many distinct rays, and that ray directions are random. There is no clean theory leading to a direct extension of Eq. (1) to the classically chaotic case, but Berry conjectured[4] that Eq. (1) still applies, at least qualitatively, with the added reasonable assumption that each time a ray returns to a given region, its history since the last visit generates a random phase. That is, $S_n(\mathbf{x})$ is not in phase with $S_{n+1}(\mathbf{x})$, etc. Locally, the eigenfunction is a superposition of an unlimited number of plane waves of fixed wave-vector magnitude but random amplitude, phase, and direction:

$$\Psi^{\text{rand}}(\mathbf{x}) = \sum_n a_n \exp(i\mathbf{k}_n \cdot \mathbf{x}), \qquad (2)$$

where a_n includes a random phase factor. By the central limit theorem, $\Psi^{\text{rand}}(\mathbf{x})$ is then Gaussian random. Berry showed that it has the coordinate-space correlation function (for the two-dimensional case)

$$C(\mathbf{x}, \mathbf{x}+\boldsymbol{\delta}) = \int \Psi^{\text{rand}*}(\mathbf{x}) \Psi^{\text{rand}}(\mathbf{x}+\boldsymbol{\delta}) d^3x = \text{const} \times J_0(k\delta), \qquad (3)$$

where J_0 is the Bessel function of zero order and where k and δ are the magnitudes of $\mathbf{k}_n$ and $\boldsymbol{\delta}$.

It seems desirable to know what a random eigenfunction actually looks like. Although technically the characterization of $\langle \mathbf{x} | \Psi^{\text{rand}} \rangle$ as "Gaussian random with Bessel-function spatial correlation" is a fairly complete statement about the ideally random eigenfunction, it does not give us a good idea of the appearance of such a function. In the literature, comparisons of $\langle \mathbf{x} | \Psi^{\text{rand}} \rangle$ with Gaussian random speckle patterns (as in scattered laser light) have been made. These analogies are not completely correct, as we show here.

It is easy to generate explicitly a random superposition of plane waves with equal wave-vector magnitudes and to plot the result in coordinate space. Apparently, this has not been done previously. We were led to to it by the realization that certain unusual structures should exist in coordinate-space plots of the kind of random wave function appropriate to semiclassical studies.

The existence of these structures can be discovered by investigation of another correlation function, involving two coherent states instead of two positions. Consider first the overlap between a coherent state $|\alpha\rangle$ and a particular random-wave superposition $|\Psi^{\text{rand}}\rangle$. We choose the coherent state to have the same average wave-vector magnitude as the fixed k used in the plane-wave superposition by requiring that $|\mathbf{p}_\alpha|^2=\hbar^2k^2$, where $\mathbf{p}_\alpha$ is $\langle\alpha|\hat{\mathbf{p}}|\alpha\rangle$. In coordinate space, the coherent state (Gaussian wave packet) reads

$$\langle\mathbf{x}|\alpha\rangle\equiv\gamma_{x_\alpha p_\alpha}(\mathbf{x})$$
$$=\exp[-A(\mathbf{x}-\mathbf{x}_\alpha)^2/2\hbar+i\mathbf{p}_\alpha\cdot(\mathbf{x}-\mathbf{x}_\alpha)/\hbar].$$

It has average position $\mathbf{x}_\alpha$ and momentum $\mathbf{p}_\alpha$. This Gaussian is a flexible test function which balances position uncertainty and momentum uncertainty in a known way: $\Delta p\sim(\hbar A)^{1/2}$, $\Delta x\sim(\hbar/A)^{1/2}$. The Gaussian includes the previous spatial correlation as the limit $A\to\infty$, but this limit corresponds to complete uncertainty of the magnitude and direction of the momentum.

The statistics of the amplitude $\langle\alpha|\Psi^{\text{rand}}\rangle$ is Gaussian random if we examine a large number of positions $\mathbf{x}_\alpha=\langle\alpha|\hat{\mathbf{x}}|\alpha\rangle$ and directions $\mathbf{p}_\alpha$ of the coherent state. Suppose that by chance we had picked an $|\alpha\rangle$ with a very large overlap with $|\Psi^{\text{rand}}\rangle$. Now, we evolve this coherent state for a short time by use of the free-particle propagator. It moves in the direction of $\mathbf{p}_\alpha$ a distance $|\mathbf{p}_\alpha|t/m$, and spreads a little. If the picture of $\langle\mathbf{x}|\Psi^{\text{rand}}\rangle$ as being a speckle pattern were correct, then our choice of $|\alpha\rangle$ would correspond to our having fortuitously placed the wave packet on the top of one of the spikes in the speckle pattern. Time evolution of $|\alpha\rangle$ would result in motion away from the spike and a rapid decrease in the packet's overlap with $|\Psi^{\text{rand}}\rangle$. We find, however, that the magnitude of the overlap must remain unchanged,

$$|\langle\Psi^{\text{rand}}|\exp(-iHt/\hbar)|\alpha\rangle|=|\langle\Psi^{\text{rand}}|\exp(-iEt/\hbar)|\alpha\rangle|=|\langle\Psi^{\text{rand}}|\alpha\rangle|,$$

since $|\Psi^{\text{rand}}\rangle$ is an eigenstate of our Hamiltonian, by construction. Thus, at the position $\mathbf{x}_\alpha+\mathbf{p}_\alpha t/m$, a coherent state with the same momentum also has anomalously large overlap with $|\Psi^{\text{rand}}\rangle$. If we move far enough away, however, spreading of the wave packet (the evolving coherent state) in the time necessary to get there will be significant enough to degrade the argument. This reasoning suggests that the regions of large amplitude of $\langle\mathbf{x}|\Psi^{\text{rand}}\rangle$ occur in short segments, rather than as isolated spikes. Since the momentum is aimed along the axis of these segments, the associated nodal lines will run perpendicular to them. The special feature which causes this effect, and which is not present in the usual speckle phenomenon, is the use of a fixed wave-vector magnitude in the plane waves composing $|\Psi^{\text{rand}}\rangle$.

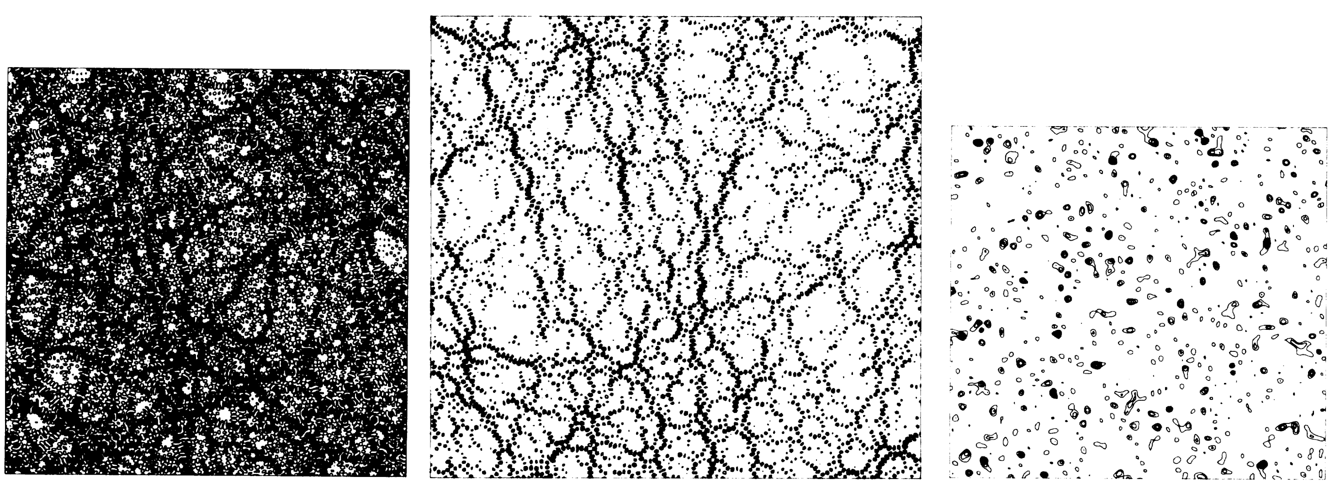

FIG. 1. (a) Contour map of $\langle\mathbf{x}|\Psi^{\text{rand}}\rangle$ over a region 60 wavelengths on a side. The function $|\Psi^{\text{rand}}\rangle$ is a superposition of 400 cosine waves, each with a random orientation $\mathbf{k}$, a random phase shift, and a Gaussian random amplitude. (b) Another view at the same scale, this time of $|\langle\mathbf{x}|\Psi^{\text{rand}}\rangle|^2$. 10000 cosine waves were used. (c) A speckle pattern, obtained by superposition of 1000 cosines with various wave-vector magnitudes.

If mixed magnitudes are used, the replacement of $\langle\Psi^{\text{rand}}|\exp(-iHt/\hbar)$ by $\langle\Psi^{\text{rand}}|\exp(-iEt/\hbar)$ is no longer correct and the argument leading to the suggestion of segments no longer applies. However, semiclassical considerations force us to use fixed magnitude at each $\mathbf{x}$.

The random pattern of ridges is obvious in Fig. 1(a), which shows a contour plot of $\langle\mathbf{x}|\Psi^{\text{rand}}\rangle$ over a region 60 wavelengths on a side. The function $|\Psi^{\text{rand}}\rangle$ is a superposition of 400 cosine waves, each with a random direction, a random phase shift, and a Gaussian random amplitude. Figure 1(b) shows another such state, this time built out of 10000 cosine waves; and the plot is of $|\langle\mathbf{x}|\Psi^{\text{rand}}\rangle|^2$. On this scale, many of the segments seem curved. This is explained by more detailed consideration of the correlation functions, which is done next. Care has been taken to insure that these states are truly "Gaussian random with Bessel-function spatial correlation." The robustness of these structures has also been checked by use of various numbers of cosine waves, various random-number generators, and a smooth, even distribution of orientations for the wave vectors for the cosine waves (but still with a random phase attached!). The resulting states *always* display this ridge network. Finally, we show for comparison a typical speckle pattern, where a range of wave-vector magnitudes is used, Fig. 1(c). The differences between Fig. 1(c) and Figs. 1(a) and 1(b) are fairly strong.

We now proceed to quantify the effects seen in the coordinate-space plots, Figs. 1(a) and 1(b), and discussed qualitatively above. First, we clean up a dusty corner of the problem: The arguments given here and in the literature have so far been in terms of superpositions of *complex* plane waves, but for systems with time-reversal symmetry, one can always taken the eigenfunctions to be real. For every plane wave $a_n\exp(i\mathbf{k}_n\cdot\mathbf{x})$ we must have another, $a_n^*\exp(-i\mathbf{k}_n\cdot\mathbf{x})$. This will affect some of the properties of the random wave functions, but the plots of $|\Psi^{\text{rand}}(\mathbf{x})|^2$ are not affected. The quasilinear structures remain true for both the real and the complex random superpositions.

The eigenfunction is to be modeled with random superpositions of phase-shifted standing cosine waves, rather than traveling plane waves. If we abbreviate the plane wave with wave-vector magnitude k traveling at an angle θ with respect to the x-axis as $|\theta\rangle$, the cosine waves take on the form $|\Psi^{\text{rand}}\rangle=\sum_j[a_j|\theta_j\rangle+a_j^*|\theta_j+\pi\rangle]$. The correlation function we choose to investigate is

$$C(\alpha,\beta)=\langle(P_\alpha^\Psi-P)(P_\beta^\Psi-P)\rangle, \tag{4}$$

where $P_\alpha^\Psi=|\langle\alpha|\Psi^{\text{rand}}\rangle|^2$, $P=\langle|\langle\alpha|\Psi^{\text{rand}}\rangle|^2\rangle=\langle|\langle\beta|\Psi^{\text{rand}}\rangle|^2\rangle$. We have chosen to examine the cross correlation of *probabilities* rather than *amplitudes* because this directly probes the correlation of large overlap of two different coherent states with a random cosine-wave superposition, eliminating the "dephasing" effects which can occur when amplitudes are considered. (In the present context we are not interested in the dephasing).

Inserting the expression for $|\Psi^{\text{rand}}\rangle$ into Eq. (4), discarding terms random in phase, and replacing sums by integrals where appropriate, we get

$$C(\alpha,\beta)=\int d\theta\int d\theta'\langle\alpha|\theta\rangle\langle\theta|\beta\rangle\langle\beta|\theta'\rangle\langle\theta'|\alpha\rangle+\int d\theta\int d\theta'\langle\alpha|\theta\rangle\langle\theta'|\beta\rangle\langle\beta|\theta+\pi\rangle\langle\theta'+\pi|\alpha\rangle \tag{5}$$

Examine a piece of this equation, namely $c_{\alpha,\beta}=\int d\theta\langle\alpha|\theta\rangle\langle\theta|\beta\rangle$. This integral is related to Bessel functions $J_n(x)$. These may be written

$$J_n(x)=\frac{i^{-n}}{2\pi}\int_0^{2\pi}\exp[ix\cos(\theta+\theta')+in(\theta+\theta')]d\theta,$$

where we have added a phase θ' to the integrand. The integral is, however, independent of θ', *even for complex* θ'; we arrive at

$$c_{\alpha,\beta}=e^{-2\hbar k^2/A}J_0(k(\mathbf{z}\cdot\mathbf{z})^{1/2}), \tag{6a}$$

where

$$\mathbf{z}=\mathbf{x}_\alpha-\mathbf{x}_\beta-i(\mathbf{p}_\alpha+\mathbf{p}_\beta)/A. \tag{6b}$$

The parameter A is the Gaussian spread parameter. After a little more algebra we obtain

$$C(\alpha,\beta)=\exp(-4\hbar k^2/A)\{|J_0(k(\mathbf{z}\cdot\mathbf{z})^{1/2})|^2+|J_0(k(\mathbf{z}'\cdot\mathbf{z}')^{1/2})|^2\}, \tag{7}$$

where $\mathbf{z}$ is as before and $\mathbf{z}'=\mathbf{x}_\alpha-\mathbf{x}_\beta-i(\mathbf{p}_\alpha-\mathbf{p}_\beta)/A$.

In the limit $A\to\infty$, the Gaussians become coordinate-space delta functions, and the correlation function $C(\alpha,\beta)$ reduces to $\simeq|J_0(k\delta)|^2$, where $\delta=|x_\alpha-x_\beta|$, in agreement with the previous coordinate-space results.

In Fig. 2(a), we plot $C(\alpha,\beta)$ for $k=2\pi$, $A=1$, as a function of x and y, for the geometry shown in Fig. 2(b). The tendency for the overlap to be large in the direction of motion (x in this case) is evident. The manifestation of this in coordinate-space plots of $|\Psi^{\text{rand}}\rangle$ is the ridgelike structures seen in Fig. 1(a).

The arclike structures seen in Fig. 1(b), which surround regions of low amplitude, can be understood as follows. Imagine placing a Gaussian coherent state $|\alpha\rangle$ in one of these low-amplitude regions, with some direction $\mathbf{k}$. The magni-

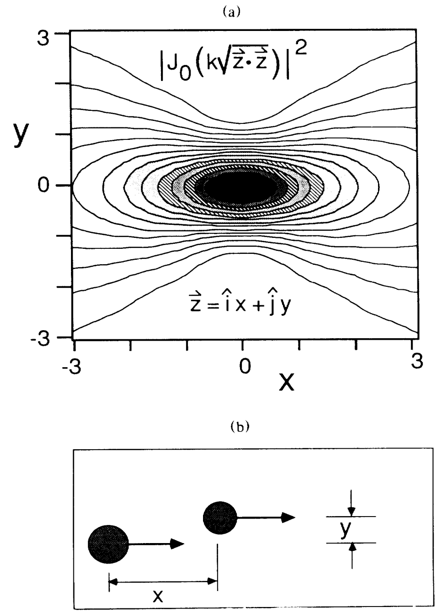

FIG. 2. (a) $C(\alpha,\beta)$ for $k=2\pi$, $A=1$, as a function of x and y, for the geometry shown in (b). The axes are labeled in wavelengths. The momenta of the two packets were chosen to be parallel.

tude of the overlap, which is small, must remain the same as $|\alpha\rangle$ is propagated into the surrounding region, which has regions of much larger amplitude $|\langle \mathbf{x}|\Psi^{\text{rand}}\rangle|$. Only if the nodal structure is quite different can the magnitude remain small. The propagating packet has its nodal lines perpendicular to **k**, and note that the nodes of the semicircular regions surrounding the low-amplitude domains are approximately radial from the center of the domain. This contrasting nodal structure guarantees that the overlap $|\langle\alpha(t)|\Psi^{\text{rand}}\rangle|$ will remain small.

Finally, we speculate on the significance of the quasilinear structures seen in Fig. 1 and evidenced in $C(\alpha,\beta)$. These structures are clearly disorganized in Figs. 1(a) and 1(b), but previously one of us[1] and McDonald[2] have noted a strong correlation between similar structures and unstable periodic orbits in classically chaotic Hamiltonian systems. In the case of the stadium enclosure, the structures are rather striking,[1] and bear direct correspondence to some simple and some moderately complex unstable periodic orbits. Locally, these structures, which were dubbed "scars,"[1] bear a strong resemblance to the quasilinear segments seen in Fig. 1(a). Instead of forming a random network as in Fig. 1(a), they form around certain classes of periodic orbits. A theory for the scars was given,[1] showing which periodic orbits were likely to be the loci of scars. What is needed is a more complete theory which shows how the scars arise out of the random network of Fig. 1(a).

[1]E. J. Heller, Phys. Rev. Lett. **53**, 1515 (1984).

[2]S. W. McDonald and A. N. Kaufman, Phys. Rev. Lett. **42**, 1189 (1979); S. W. McDonald, Ph.D. thesis, Lawrence Berkeley Laboratory Report No. LBL 14837, 1983 (unpublished).

[3]M. Tabor, Physica (Amsterdam) **6D**, 195 (1983); M. Shapiro and G. Goelman, Phys. Rev. Lett. **29**, 1714 (1984); G. Hose and H. S. Taylor, Phys. Rev. Lett. **51**, 947 (1983); B. V. Chirikov, "An Example of Chaotic Eigenstates in a Complex Atom" (to be published); M. V. Berry and M. Robnik, J. Phys. A **19**, 1365 (1986); T. H. Seligman, J. J. M. Verbaarschot, and H. A. Weidenmüller, Phys. Lett. **167B**, 365 (1986); M. Shapiro, R. D. Taylor, and P. Brumer, Chem. Phys. Lett. **106**, 325 (1984); D. L. Shepelyansky, Phys. Rev. Lett. **56**, 677 (1986); B. Eckhardt, "Exact Eigenfunctions for a Quantized Map" (to be published).

[4]M. V. Berry, in *Chaotic Behavior of Deterministic Systems,* edited by G. Looss, R. Helleman, and R. Stora (North-Holland, New York, 1983), p. 171; S. Fishman, D. R. Grempel, and R. E. Prange, Phys. Rev. Lett. **49**, 509 (1982); R. Balian and C. Bloch, Ann. Phys. (N.Y.) **69**, 76 (1972); G. Casati, I. Guarneri, and F. Valz-Gris, Phys. Rev. A **3**, 1586 (1984); M. Robnik, J. Phys. A **17**, 1049 (1984); M. V. Berry and M. Robnik, J. Phys. A **17**, 2413 (1984); P. Pechukas, Phys. Rev. Lett. **51**, 943 (1983); M. C. Gutzwiller, Physica (Amsterdam) **5D**, 183 (1982).