

ϵ Expansions for Crumpled Manifolds

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To study properties of self-avoiding surfaces, we consider D -dimensional elastic manifolds embedded in d -dimensional space. In the (d, D) plane there is a line separating ideal (Gaussian) and self-avoiding (interacting) behaviors. We develop the techniques for systematic perturbation expansions, and renormalization-group analysis, about *any point* on this line. The usual ϵ expansion for polymers about $d=4$ is a special case. The exponents ν and γ are calculated to lowest order, providing very good values for polymers ($D=1$), and reasonable indications for surfaces ($D=2$).

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One current challenge in theoretical physics is to understand the behavior of surfaces.¹ In particle physics this question arises in both string and gauge theories: World sheets of strings are two-dimensional manifolds, while random surfaces appear in high-temperature expansions of lattice gauge theories. In condensed matter physics, random surfaces are essential to the understanding of microemulsions, and appear naturally in chalcogenide glasses, lipid bilayers, and sheet polymers.

In the absence of self-avoidance, essentially all models of linear polymer chains lead to a harmonic theory at long wavelengths²; i.e., a continuum free energy proportional to $\int dx (d\mathbf{r}/dx)^2$, where $\mathbf{r}(x)$ is the monomer position at a distance x along the chain. Surfaces, on the other hand, fall into a variety of universality classes,¹ depending on factors such as rigidity, surface tension, and/or local microscopic constraints. A particularly simple universality class is exemplified by "tethered surfaces," composed of particles with a fixed two-dimensional connectivity.³ Here, entropic effects generate a harmonic elastic free energy F , which is a simple generalization of the polymer one, $F \sim \int d^2x (\nabla \mathbf{r})^2$. It is therefore natural to treat self-avoidance in tethered surfaces by use of perturbative renormalization-group (RG) methods developed for polymers. Problems arise, however, because although self-avoiding (SA) interactions are irrelevant for polymers in dimensionalities $d > 4$, allowing an expansion about four dimensions,⁴ these interactions are always relevant for surfaces, making a similar expansion impossible.

One way around this difficulty is to consider general D -dimensional SA manifolds. The simplest continuum energy interpolating between polymers and elastic surfaces is proportional to $\int d^Dx (\nabla \mathbf{r})^2$. Dimensional analysis for this free energy yields $[r] \sim [x]^{1-D/2}$, corresponding to an *ideal* fractal dimension of $d_f^0 = 2D/(2-D)$, which interpolates between 2 for random

walks and ∞ for random surfaces. Two such ideal manifolds (or two widely separated segments of the same manifold) intersect only if embedded in a space of dimension $d < 2d_f^0$. There is thus a critical dimensionality $d^*(D) = 2d_f^0 = 4D/(2-D)$, separating ideal and SA behaviors as depicted in Fig. 1.

The critical line, of course, passes through the expansion point ($d^* = 4, D^* = 1$) for polymers, but in fact *any point* on this line is an equally good expansion candidate. For example, we can stay in three dimensions ($d=3$), and change the manifold dimensionality D . Self-avoidance dominates for solid elastic cubes ($D=3$), but is less important for elastic surfaces ($D=2$). It produces relatively small corrections to the Gaussian result for linear manifolds ($D=1$), and becomes negligible when $D < D^* = \frac{6}{7}$. An $\epsilon' = D - \frac{6}{7}$ expansion corresponds to

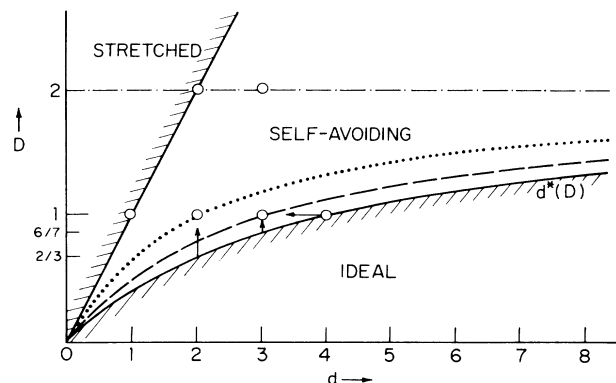


FIG. 1. Scaling regimes for manifolds. SA interactions became relevant at an "upper critical boundary" $d^*(D)$ and saturate at the $d=D$ "lower critical boundary" (solid lines). Onset of relevance of three-point and ∞ -point interactions is indicated by dashed and dotted lines, respectively. The ϵ expansions quoted in the text are indicated by arrows.

SA polymers when $\epsilon' = \frac{1}{7}$, and SA surfaces when $\epsilon' = \frac{8}{7}$. For $d=2$ the critical manifold dimensionality is $D^* = \frac{2}{3}$. In this paper, we generalize the RG techniques of Ref. 4 to all points along the critical line. The lowest-order results for the exponents ν and γ provide new, more precise exponents for polymers ($D=1$), and encouraging indica-

tions of the trends for surfaces ($D=2$).

Model.—The manifold configurations are described by $\mathbf{r}(\mathbf{x})$, where \mathbf{x} is a D -dimensional vector labeling positions within the *internal* space of the manifold, while \mathbf{r} is a d -dimensional vector describing the embedding of the particle in *external* space. The partition function is obtained from

$$Z = \int \mathcal{D}\mathbf{r}(\mathbf{x}) \exp \left\{ -\frac{K}{2} \int d^D x (\nabla \mathbf{r})^2 - \frac{v}{2} \int d^D x d^D x' \delta^d(\mathbf{r}(\mathbf{x}) - \mathbf{r}(\mathbf{x}')) \right\}, \quad (1)$$

where $(\nabla \mathbf{r})^2 = \sum_{a=1}^D (\partial \mathbf{r} / \partial x_a)^2$. The first term in the action describes the ideal elastic manifold, while the second term is an SA repulsion on contact. Equation (1) for $D=1$ is the Edwards model of polymers,^{2,4} while for $D=2$ it corresponds to tethered surfaces.³ Under a rescaling $\mathbf{x} \rightarrow \lambda \mathbf{x}$, and $\mathbf{r} \rightarrow \lambda^{\nu} \mathbf{r}$, Eq. (1) gives $K \rightarrow \lambda^{y_K} K$ and $v \rightarrow \lambda^{y_v} v$ with $y_K = 2\nu + D - 2$ and $y_v = 2D - d\nu$. The “ideal” fixed point corresponds to $\nu_0 = 1 - D/2$, which leaves the elastic term invariant, with $y_v^0 = 2D - (2 - D)d/2$. In agreement with the previous geometrical argument, we find $y_v^0 > 0$ and SA interactions relevant for $d < d^*(D) = 2d^0$. An approximate Flory argument^{2,3} in the SA regime requires the two terms to scale the same way ($y_K = y_v$) and leads to the exponent $\nu_F = (D+2)/(d+2)$ for the radius of gyration. We expect fully stretched manifolds ($\nu=1$) when $d \leq D$ (see Fig. 1).

A prerequisite of a *systematic* RG study of the SA regime is understanding the relevance of higher-order interactions. An n -particle interaction is represented by a term

$$u_n \int \prod_{i=1}^n d^D x_i \prod_{i=1}^{n-1} \delta^d(\mathbf{r}(\mathbf{x}_i) - \mathbf{r}(\mathbf{x}_{i+1}))$$

in the free energy. At the ideal fixed point the scaling of this interaction is governed by $y_n^0 = nD - (n-1)$

$\times (2 - D)d/2$, which becomes important at the *Gaussian* fixed point when $d < d_n^*(D) = n d^0(D)/(n-1)$. We expect these higher-order interactions to be irrelevant except at special multicritical points.² The critical curve for three-body interactions is depicted by the dashed line in Fig. 1, and is important to tricritical, or θ -point, behavior.² The accumulation point of these higher-order lines at $d_\infty^* = d^0(D)$ is the dotted line in Fig. 1.

Our systematic calculations are performed for a *monodisperse* solution of *hyperspherical* manifolds of internal radius X with open boundaries. (These restrictions are for the sake of simplicity, and not essential.) Our treatment follows the spirit, and as far as possible the notation, of a renormalization-group treatment of polymer solutions by des Cloizeaux.⁴ What follows is a brief outline of the calculation; details will appear in a future publication.

Perturbation Theory.—The SA interaction in Eq. (1) is treated perturbatively around the ideal manifold. The expansion is shown diagrammatically in Fig. 2, with a dashed line depicting the bare SA interaction between two points on the manifold. The rules for calculating these diagrams are as follows:

(i) The role of the propagator is taken by the two-point function [Fig. 2(a)]

$$\zeta_1(\mathbf{k}, \mathbf{x}_1; -\mathbf{k}, \mathbf{x}_2) = \frac{1}{Z_0} \int \mathcal{D}\mathbf{r}(\mathbf{x}) \mathcal{P}\{\mathbf{r}(\mathbf{x})\} \exp\{i\mathbf{k} \cdot [\mathbf{r}(\mathbf{x}_1) - \mathbf{r}(\mathbf{x}_2)]\}, \quad (2)$$

where Z_0 is the noninteracting partition function, and $\mathcal{P}\{\mathbf{r}(x)\}$ is the Boltzmann weight implicit in Eq. (1). General

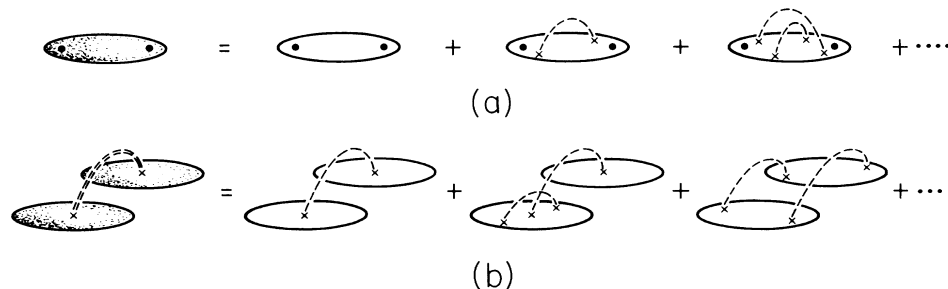


FIG. 2. Perturbation “manifold diagrams” for (a) the two-point propagator, and (b) interaction renormalization.

n -point functions for the *ideal manifold* are obtained from

$$\left\langle \exp \left[i \sum_a \mathbf{k}_a \cdot \mathbf{r}(\mathbf{x}_a) \right] \right\rangle_0 = \exp \left[- \sum_{\alpha < \beta} \frac{\mathbf{k}_\alpha \cdot \mathbf{k}_\beta}{K} C_D(\mathbf{x}_\alpha - \mathbf{x}_\beta) \right], \quad (3)$$

where $C_D(\mathbf{x}) = |\mathbf{x}|^{2-D}/S_D(2-D)$ is the Coulomb interaction in D dimensions (S_D is the area of a unit hypersphere).

(ii) The terms in the expansion are calculated by writing of $\delta^d(\mathbf{r}(\mathbf{x}) - \mathbf{r}(\mathbf{x}'))$ as $\int d^d q / (2\pi)^d \exp\{i\mathbf{q} \cdot [\mathbf{r}(\mathbf{x}) - \mathbf{r}(\mathbf{x}')]\}$, and use of Eq. (3). This is followed by integration over the magnitudes and internal positions of the charges introduced by the perturbation term. Note that the concept of reducibility with respect to the cutting of propagator lines has no meaning for general manifold diagrams [e.g., in the second-order term of Fig. 2(a)].

Regularization.—There are a number of divergences that have to be removed from the perturbation expansion. One set is related to the limit $|\mathbf{x}_1 - \mathbf{x}_2|/\alpha \rightarrow \infty$ (α is the short-distance cutoff of the model), and is removed by mass subtractions. The more important divergences result from the relevance of v as

$$\varepsilon \equiv 4D - (2-D)d$$

tends to zero, and are regularized by continuation to noninteger values of ε . (Note that, for $d=3, \varepsilon=7\varepsilon'$.) There are three important quantities whose expansions need to be regularized at the lowest order.

(i) The most complex behavior is exhibited by the partition function. It is sufficient to examine $\zeta_1(\mathbf{k}, \mathbf{x}_1; -\mathbf{k}, \mathbf{x}_2)$, which for $\mathbf{k}=0$ reduces to [see Eq. (2)] the ratio of noninteracting and interacting partition functions (denoted by ζ_1). The lowest-order correction from Fig. 2(a) is then

$$-\frac{v}{2} \int d^D x d^D x' \int \frac{d^d q}{(2\pi)^d} \exp \left[-\frac{q^2}{K} C_D(\mathbf{x} - \mathbf{x}') \right]. \quad (4)$$

By a change to relative and center-of-mass coordinates, it can be seen that the main contribution from Eq. (4) comes from the manifold "volume" and goes as X^D . There are additional divergences from the manifold edges, corners, etc., which behave as X^{D-1}, X^{D-2}, \dots . Finally, there is a contribution independent of the center-of-mass coordinate that becomes logarithmically divergent at $\varepsilon=0$, and implies that the partition function Z scales with the internal manifold size X as

$$Z \sim \exp[f_D X^D + f_{D-1} X^{D-1} + \dots] X^\gamma. \quad (5)$$

We expect f_D, f_{D-1}, \dots to be nonuniversal and shape dependent, while γ is universal. Numerical simulations of surfaces have to be fitted by Eq. (5), and f_2 and f_1 identified, before γ is estimated. Analogous subtractions of X^D, X^{D-1}, \dots terms are required in a perturbative calculation of γ . Here D such subtractions have to be car-

ried out in a manner that can be continued in D . In fact the integrals in Eq. (4) lead to a regularized partition function $\zeta_1 = 1 + 2\mathcal{A}(D)(z/\varepsilon)$, where

$$z = \frac{S_D^2}{D} \left[\frac{K S_D (2-D)}{4\pi} \right]^{2D/(2-D)} X^{\varepsilon/2v}$$

is the effective coupling constant. $\mathcal{A}(D)$ is the constant term in the expansion of $-[t/(1-t^2)]^D$ about $t=1$, obtained after removal of the D leading divergences. (For nonintegral D , $\mathcal{A}(D)$ can be evaluated from a power series.)

(ii) Relative fluctuations of two points on the manifold are obtained from the $O(k^2)$ term in $\zeta_1(\mathbf{k}, x_1; -\mathbf{k}, x_2)$. This is a straightforward generalization of our previous calculation³ of the correction to size fluctuations of surfaces. The final result is

$$\begin{aligned} \langle |\mathbf{r}(\mathbf{x}_1) - \mathbf{r}(\mathbf{x}_2)|^2 \rangle \\ = \frac{2d}{K} C_D(\mathbf{x}_1 - \mathbf{x}_2) \left[1 + \frac{2-D}{2} \left(\frac{z}{\varepsilon} \right) \right]. \end{aligned} \quad (6)$$

(iii) Renormalization of the interaction is studied via the second virial coefficient, whose perturbation series involves connected diagrams of two manifolds as in Fig. 2(b). Of the two $O(v^2)$ terms in Fig. 2(b), one is interaction reducible, and obtained simply as the product of v and ζ_1 (thus interaction reducibility is still useful for manifolds). The other I -irreducible part involves a third integral, and eventually leads to the regularized form

$$\zeta_2 = -2v \left[\frac{S_D X^D}{D} \right]^2 \left[1 + 4\mathcal{A}(D) \frac{z}{\varepsilon} - 2\mathcal{C}(D) \frac{z}{\varepsilon} \right], \quad (7)$$

with

$$\mathcal{C}(D) = \frac{\pi^{1/2} \Gamma(2/(2-D))}{2^{2D/(2-D)} \Gamma((2+D)/2(2-D))}.$$

Renormalization.—Equations (5)–(7) contain all that is necessary to calculate the scaling exponents to $O(\varepsilon)$, since $\zeta_1 \sim X^{\gamma-1}$ and $\langle |\mathbf{r}(\mathbf{x}_1) - \mathbf{r}(\mathbf{x}_2)|^2 \rangle \sim |\mathbf{x}_1 - \mathbf{x}_2|^{2\gamma}$ (i.e., the manifold size R scales as X^γ). The fixed-point value z^* is obtained by our requiring the second virial coefficient ζ_2/ζ_1^2 to scale as R^d . Generalization of the polymer calculations of Ref. 4 then leads to the lowest-order exponents

$$\begin{aligned} v &= \frac{2-D}{2} + \frac{2-D^*}{8[D^* + 2\mathcal{C}(D^*)]} \varepsilon, \\ \gamma &= 1 + \frac{\mathcal{A}(D^*)}{D^* + 2\mathcal{C}(D^*)} \varepsilon. \end{aligned} \quad (8)$$

Note that each point in the SA regime of Fig. 1 corresponds to a given ε , where scaling exponents can be obtained by expansion about any point (d^*, D^*) on the critical line.

For example, an expansion about $d^*=3$ gives $\nu = \nu_0 + 0.067\varepsilon$ [$\nu_0 = (2-D)/2$] and $\gamma = 1 + 0.122\varepsilon$, which can be used for polymers ($\varepsilon' = \frac{1}{7}$), and surfaces ($\varepsilon' = \frac{8}{7}$) in three dimensions. The polymer results $\nu = 0.567$ and $\gamma = 1.122$ are very good, but the surface exponent of $\nu = 0.536$ is certainly incorrect as it should be greater than $\frac{2}{3}$, and the best estimate is $\nu = 0.8$.³ At present, there are no independent estimates of γ , and our result of $\gamma = 1.976$ is not unreasonable. Similarly for $d^*=2$, we have $\nu = \nu_0 + 0.075\varepsilon$, and $\gamma = 1 + 0.113\varepsilon$, leading to $\nu = 0.650$ and $\gamma = 1.226$ for two-dimensional polymers ($\varepsilon = 2$).

We have demonstrated that scaling properties of general D -dimensional manifolds can be studied systematically, generalizing previous results for polymers. The encouraging lowest-order results for $D=1$ suggest future applications of this technique for polymers. The results for surfaces ($D=2$), however, although indicating the qualitative trends, are not of much quantitative value. Going to higher orders, searching for optimal expansion points, or combining with $d \rightarrow 0$ and/or $d \rightarrow \infty$ series are possible avenues of further research.

At a more fundamental level this calculation can be regarded as generalizing traditional techniques in field theory. Figure 2 depicts Feynman diagrams in which the usual propagator lines are replaced by manifolds. The rules and organizations of these manifold diagrams as

outlined above are in many ways distinct from Feynman diagrams. From this point of view the SA-manifold calculations serve as a prototype for a variety of new perturbation studies—e.g., to calculate critical behavior of $O(n)$ spin models, which is one of the directions we are currently pursuing.

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Note added.—After this work was submitted for publication we learned of work in progress along similar lines by J. A. Aronovitz and T. C. Lubensky. Dr. Aronovitz kindly pointed out an error in our initial evaluation of the formula for ν .

¹For a general review, see J. Fröhlich, in *Applications of Field Theory to Statistical Mechanics*, edited by L. Garido, Lecture Notes in Physics Vol. 216 (Springer, Berlin, 1985).

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