Path-Integral Approach to Ising Spin-Glass Dynamics

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Path integrals are introduced for Glauber dynamics of interacting Ising spins, equivalent to stochastic equations for local magnetizations. This enables us to perform quenched averages. For the Sherrington-Kirkpatrick model of spin-glasses the low-frequency dynamics is determined for all temperatures and fields.

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Spin-glasses are essentially a dynamic phenomenon.[1] Now there is numerical evidence for a true phase transition of Ising spin-glasses in three dimensions.[2] The transition in an external field is under discussion.[3] In the ordered phase one assumes the existence of infinite energy barriers corresponding to infinite relaxation times. Relaxational dynamics of Ising spins is conveniently described by Glauber dynamics[4] which successfully has been used in Monte Carlo simulations of Ising spin-glasses. In this Letter I propose a path-integral method to find analytic results for quenched random Ising systems. I apply it to the Sherrington-Kirkpatrick (SK) model of spin-glasses.[5] The method may be useful for the random-field model as well, which has interesting dynamic properties too.[6]

Sompolinsky and Zippelius[7] have used soft-spin dynamics to derive results for Ising spin-glasses, especially for the SK model with long-range random interactions ($\langle J_{ij}^2\rangle_{\text{Av}}=J^2/N$). However, one is only able to formulate the problem perturbationally with respect to the four-spin interaction of the soft spins, u, which goes to infinity in the Ising limit. Fortunately, u drops out in certain results in the critical regime, so that one hopes that these results will be correct for "hard spins" too. One is not able to calculate any dynamic correlation function of soft spins in a wider frequency and temperature regime, not even for free spins. For free Glauber spins, however, one knows all correlation functions, which are useful to construct a field theory.[8] Below I present a generating functional for all correlation and response functions.

I also derive the corresponding generating functional for the SK model, which allows us to find a numerically solvable equation for the dynamic local susceptibility, valid in the low-frequency regime for all temperatures and external fields. This means that the solution shows the correct frequency behavior, while the absolute frequency scale can only be determined perturbationally. The approximation may be systematically improved, obeying the dynamic fluctuation-dissipation theorem (FDT) step by step. In the scaling regime I recover the dynamic critical exponent found by Sompolinsky and Zippelius[7] and the scaling function above the de Almeida–Thouless (AT) line, derived by Sommers and Fischer.[9] In zero external field for $T>T_c$ the equation is solved explicitly. It is correct up to $(T_c/T)^5$ for all frequencies and has a $\omega^{1/2}$ dependence for $T=T_c$. These results improve previous attempts to treat the Glauber dynamics for spin-glasses.[10] Those authors tried to introduce phenomenologically the Onsager correction term into Glauber mean-field equations.

I am also able to determine the statics or "dynamics on infinite time scales"[11] below T_c. It turns out that with use of the FDT the statics is determined by a quenched local-field distribution,[12,13] $P(y)$, so that the average magnetization is given by

$$\langle\overline{m}\rangle_{\text{Av}}=\int dy\,P(y)\tanh(\beta b+\beta y), \tag{1}$$

where b is the local external field. Similar expressions hold for local static responses. In the paramagnetic phase $P(y)$ is Gaussian with zero mean and variance J^2q^5, where $q=\langle\overline{m}^2\rangle_{\text{Av}}$ is the Edwards-Anderson order parameter ($q=0$ in zero external field). In the spin-glass phase $P(y)$ is non-Gaussian and a complicated functional of the Parisi order parameter.[14] Going back to Thouless-Anderson-Palmer (TAP) mean-field equations[15] this means that the contributions from different sites to the mean field at a given site are not statistically independent in the spin-glass phase. The marginal stability condition

$$1-(\beta J)^2\langle(1-m^2)^2\rangle_{\text{Av}}=0 \tag{2}$$

leads to the algebraic decay of correlation functions found by Sompolinsky and Zippelius (SZ).[7] $P(y)$ may be found from a solution of the SZ saddle-point equations which violate the FDT on infinite time scales. This is not a contradiction because I am considering the relaxation of the system from a nonequilibrium initial condition. Since I allow for an anomalous response on infinite time scales, the field distribution evolves from a Gaussian to $P(y)$, which is highly correlated.[13] One is also able to calculate higher-order correlation functions on infinite time scales,[16] which show the same ultrametric topology in time, as found by Mezard *et al.*[17] in the space of pure states. In fact, the method to be described consists of writing the true Glauber spin-distribution function for all times as an average over a distribution of independent spins with magnetizations obeying stochastic equations. On long time scales this corresponds to the decomposition of the Gibbs state into pure states, as suggested by De Dominicis and Young[18] and Parisi.[19]

I now sketch the derivation of the results. I consider Glauber dynamics[4] for random Ising-spin systems $\{\sigma_i\}$. Introducing the local field

$$h_i = b_i + \sum_j J_{ij}\sigma_j, \tag{3}$$

where b_i is the local external field and J_{ij} the exchange interaction, we have the master equation for the spin distribution,

$$\partial P\{\sigma,t\}/\partial t = -\sum_i (1-S_i)\tfrac{1}{2}\Gamma(1-\sigma_i\tanh\beta h_i)P\{\sigma,t\} = \mathcal{L}P\{\sigma,t\}. \tag{4}$$

I only consider one-spin–flip processes: $S_i\sigma_i = -\sigma_i$. The flip rate Γ is choosen to be constant but in general may depend on h_i. I choose the initial condition $P\{\sigma,0\} = \prod_j (1+m_j^0\sigma_j)/2$. The equilibrium solution $P\{\sigma,t\} \propto \exp(-\beta H)$ obeys detailed balance and leads to fluctuation-dissipation theorems. One may derive a number of generalized FDT's in external field.[20] Below I argue that $P\{\sigma,t\}$ may be written as a functional integral over a distribution of independent spins:

$$P\{\tilde\sigma,t\} = \int \prod_j \mathcal{D}\sigma_j \mathcal{D}\hat\sigma_j \exp\left[-\int_0^t d\tau \sum_j i\hat\sigma_j(\tau)[\sigma_j(\tau)-m_j(\tau)]\right]\prod_j[1+m_j(t)\tilde\sigma_j]/2 = \langle\prod_j[1+m_j(t)\tilde\sigma_j]/2\rangle. \tag{5}$$

The local magnetizations $m_i(t)$ obey exact stochastic equations

$$\dot m_i = i\hat\sigma_i(1-m_i^2) - \Gamma[m_i - \tanh(\beta b_i + \sum_j \beta J_{ij}\sigma_j)], \quad m_i(0) = m_i^0, \tag{6}$$

which are reminiscent of time-dependent generalizations of TAP equations. The weight $\langle\cdots\rangle$ determines general spin correlations $\langle\sigma_1(t_1)\cdots\sigma_n(t_n)\rangle$ $(0<t_i<t)$. For example, for independent spins

$$\langle(\sigma_1(t_1)\cdots\sigma_n(t_n)\rangle = \frac{\delta}{\delta i\hat\sigma_1(t_1)}\cdots\frac{\delta}{\delta i\hat\sigma_n(t_n)}\exp\left[\sum_j\int_0^t d\tau\, i\hat\sigma_j m_j\right]\Bigg|_{\hat\sigma_i=0}. \tag{7}$$

The distribution $P\{\tilde\sigma,t\}$ can be averaged over the weighting function

$$P(J_{ij}) \propto \exp(-J_{ij}^2 z/2J^2) \tag{8}$$

(SK: $z=N$) using the identity

$$1 = \prod_j \int \mathcal{D}h_j \prod_\tau \delta\left[h_j(\tau) - b_j - \sum_i J_{ji}\sigma_i(\tau)\right]. \tag{9}$$

Applying this identity directly to the formal solution $P\{\sigma,t\} = T\exp[\int_0^t d\tau \mathcal{L}(\tau)]P\{\sigma,0\}$ and introducing conjugate variables $\hat h_j$ for the exponential representation of the δ function we may evaluate the time-ordering product T leading to Eq. (5).

For the SK model we can perform the quenched average by saddle-point integration for $N\to\infty$ following SZ.[7] This leads to mean-field equations for local correlation and response functions

$$C(t,t') = \langle\sigma(t)\sigma(t')\rangle, \quad G(t,t') = \delta\langle\sigma(t)\rangle/\delta b(t'). \tag{10}$$

With an obvious abbreviation for the time arguments, $\langle X\rangle$ is given by

$$\langle X\rangle = \int \mathcal{D}h\,\mathcal{D}\hat h\,\mathcal{D}\sigma\,\mathcal{D}\hat\sigma \exp\left[-i\int \hat h(h-b) - i\int\hat\sigma(\sigma-m)\right]\exp\left[-J^2\int C(1,2)\hat h(1)\hat h(2)/2 + J^2\int_{1>2} G(1,2)i\hat h(1)\sigma(2)\right]X, \tag{11}$$

where $m(\tau)$ obeys the stochastic differenial equation

$$\dot m = i\hat\sigma(1-m^2) - \Gamma(m - \tanh\beta h), \quad m(0) = m^0. \tag{12}$$

I have choosen a homogeneous initial condition. The $\mathcal{D}h\,\mathcal{D}\hat h$ integration leads to a local field,

$$h(1) = b + J\phi(1) + J^2\int_{1>2} G(1,2)\sigma(2), \tag{13}$$

with Gaussian correlations of ϕ, $\langle\phi(1)\rangle = 0$, $\langle\phi(1)\phi(2)\rangle = C(1,2)$. The saddle point is stable, since the solution is unique.

I first discuss the solution in the paramagnetic region. I let the initial time t_0 (which was set equal to zero) go to $-\infty$. Then the correlation functions are expected to depend only on time differences. From (11) we find the generating functional for correlation and response functions

$$Z\{i\hat{\sigma},b\}=\int dy\,P(y)\exp\left[\frac{J^2}{2}\int\tilde{C}(1,2)\frac{\delta}{\delta h(1)}\frac{\delta}{\delta h(2)}+J^2\int_{1>2}\tilde{G}(1,2)\frac{\delta}{\delta h(1)}\frac{\delta}{\delta i\hat{\sigma}(2)}\right]\exp\left[\int i\hat{\sigma}m\right]\Bigg|_{h=b+y}. \quad (14)$$

The correlation and response functions are evaluated by the appropriate derivatives of Z taken at $\hat{\sigma}=0$. We have split off the long-time parts: $C(t-t')=\tilde{C}(t-t')+q$, $G=\tilde{G}$, leading to a quenched Gaussian local field ($\langle y^2\rangle_{\text{Av}}=J^2q, \langle y\rangle_{\text{Av}}=0$). By means of the FDT one now may derive that the statics is independent of the short-time parts, $\tilde{C}$, $\tilde{G}$, leading to Eq. (1). Therefore, one knows the exact static limits of all the correlation functions and one is able to make perturbation theory for fixed $P(y)$ with respect to $J^2\tilde{C}$, $J^2\tilde{G}$, obeying the FDT step by step. This should give at least the dominant low-frequency behavior. Indeed we will see that for $|\omega|\ll\Gamma$ and fixed $P(y)$ the expansion in $(\beta J)^2$ is at the same time an expansion in $\tilde{G}(\omega)-\tilde{G}(\omega=0)$. To zero order one finds

$$\tilde{C}(t)\langle(1-m^2)\rangle_{\text{Av}}e^{-\Gamma|t|},\quad \tilde{G}(\omega)=\beta\langle(1-m^2)\rangle_{\text{Av}}[1/(1-i\omega/\Gamma)], \quad (15)$$

where $m=\tanh(\beta b+\beta y)$ and $\tilde{G}(\omega)$ is the Fourier transform of $\tilde{G}(t)$ ($=0$ for $t<0$). The next order yields

$$\tilde{G}(\omega)=\frac{\beta\langle(1-m^2)\rangle_{\text{Av}}}{1-i\omega/\Gamma}+\frac{1}{1-i\omega/\Gamma}(\beta J)^2\langle(1-m^2)^2\rangle_{\text{Av}}\left[\frac{\tilde{G}(\omega)}{1-i\omega/\Gamma}-\tilde{G}(\omega=0)\right], \quad (16)$$

with the solution

$$\tilde{G}(\omega)=\beta\langle(1-m^2)\rangle_{\text{Av}}\frac{1-(\beta J)^2\langle(1-m^2)^2\rangle_{\text{Av}}}{1-i\omega/\Gamma-(\beta J)^2\langle(1-m^2)^2\rangle_{\text{Av}}/(1-i\omega/\Gamma)}. \quad (17)$$

The validity of this approximation is violated for $\omega\neq 0$ near the AT line, defined by condition (2). In this case we have to go one step further in the expansion. Using a systematic diagram expansion we find after some algebra,

$$\begin{aligned}(1-i\omega/\Gamma)\frac{\tilde{G}(\omega)}{\beta}&=\langle(1-m^2)\rangle_{\text{Av}}+(\beta J)^2\langle(1-m^2)^2\rangle_{\text{Av}}\left[\frac{\tilde{G}(\omega)}{\beta(1-i\omega/\Gamma)}-\frac{\tilde{G}(\omega=0)}{\beta}\right]\\&\quad+(\beta J)^4\langle(1-m^2)^2\rangle_{\text{Av}}\left[\frac{\tilde{G}(\omega)}{\beta(1-i\omega/\Gamma)}-\frac{\tilde{G}(\omega=0)}{\beta}\right]^2\\&\quad+(\beta J)^4\langle 2m^2(1-m^2)^2\rangle_{\text{Av}}\left[\frac{-\int_0^\infty d\tau\, e^{i\omega\tau}\partial\tilde{C}(\tau)^2/\partial\tau}{1-i\omega/\Gamma}-\left(\frac{\tilde{G}(\omega=0)}{\beta}\right)^2\right]. \quad (18)\end{aligned}$$

This equation goes in the limit $|\omega|\ll\Gamma$ exactly to the scaling equation derived by Sommers and Fischer[9] with the help of soft-spin dynamics. A more rigorous proof of Eq. (18) in the critical regime goes along the lines of SZ.[7] Equation (18) for $|\omega|\ll\Gamma$ follows by expansion of the full renormalized $G(\omega)$ with respect to the singular parts of the self-energy, since (18) already contains the exact static vertices. Equation (18) determines also approximately the absolute frequency scale which was open so far. Note, that the FDT relates $\tilde{G}(\tau)$ and $\tilde{C}(\tau)$ by $\tilde{G}(\tau)=-\beta\dot{\tilde{C}}(\tau)$ for $\tau>0$. For all frequencies $\tilde{G}(\omega)/\beta$ is determined exactly by Eq. (18) up to order $(\beta J)^4$. The short-time behavior $\omega\to\infty$ may be determined exactly with the help of the true local-field distribution.[21] For zero external field we may set $m=0$ and solve (18) explicitly. It changes continuously from the scaling form[9] near $T=J=T_c$ to the high-temperature solution (15) or (17). The marginal stability condition (2) on the AT line leads to a power-law behavior[7] $\tilde{G}(\omega)-\tilde{G}(\omega=0)\propto(-i\omega/\Gamma)^{\nu}$. ν equals $\frac{1}{2}$ in zero external field and changes because of the last term in Eq. (18) along the AT line. It should be stressed that ν is identical to the SZ result.[7,9] Equation (18) is valid in the spin-glass phase too, if $P(y)$ is reinterpreted as mentioned in the introduction. Again we find the same power-law behavior as SZ. The nature of the spin-glass phase will be discussed below. For $T\to 0$ Eq. (18) yields, for all ω, $\tilde{G}(\omega)\propto T$.

The spin-glass phase is nonergodic. This means that there exists response over infinite time scales. These time scales may go to infinity according to an Arrhenius law, where the energy barrier scales with some power of the number of spins, N.[22] We know that the system approaches equilibrium if N is finite. In this case we regard Eqs. (9) and (10) as a saddle-point approximation and look for a solution where N drops out in the limit. A detailed analysis is required to check the consistency. We split off the long-time parts $C(1,2)=\tilde{C}(1,2)+Q(1,2)$, $G(1,2)=\tilde{G}(1,2)-\beta\Delta'(1,2)$, where Q and Δ' are allowed to vary on time scales T_0 with $\ln\Gamma T_0\gg 1$, while $\tilde{C}$ and $\tilde{G}$ vary on the scale $1/\Gamma$ and obey the FDT. T_0 is of the order of the equilibration time of the system. As above, for the long-time properties $\tilde{C}$ and $\tilde{G}$ do not contribute and in Eq. (12) we may let Γ formally go to infinity. The result is a mean-field theory for correlations on long-time scales. Introducing a hierarchy of infinite time scales one recovers Sompolinsky's equations, which give results equivalent to Parisi's replica solution of the SK model.[23] If one looks at finite-time correlations one sees that the coupling to the infinite-time correlations is governed by a local field with distribution $P(y)$. This shows that the generating functional (14) for finite time correlations is valid in the spin-glass phase too. The relation to Sommers-Dupont[13] is $P(y-b)=P_{SD}(1,y)$, where $P_{SD}(1,y)$ is the distribution of TAP magnetizations $\tanh\beta y$. Thus, the finite-time dynamics is confined to a single valley of the TAP free energy.

In summary, I have demonstrated that the path-integral approach to Glauber dynamics is a powerful tool in the treatment of the spin-glass problem. Results of soft-spin dynamics above the AT line are rigorously confirmed and generalized. The Parisi and Sompolinsky solution of the SK model are simply obtained by a hierarchical *Ansatz* of infinite time scales. Around this static solution the finite-time dynamics is determined perturbationally. Results may be compared with Monte Carlo computer simulations. It is straightforward to construct a field theory for short-range systems. In the future the method may be applied to kinetics of ferromagnets and dynamical optimization. An interesting application is the dynamics of neural networks, where one may allow for asymmetric bonds, i.e., violation of detailed balance. For instance, one may solve directly the fully asymmetric SK model as considered by Hertz *et al.*[24] and show that in this case a transition to a spin-glass state is forbidden.

[1]For recent reviews on the experimental and theoretical situation, see K. H. Fischer, Phys. Status Solidi b **130**, 13 (1985); K. Binder and A. P. Young, to be published.

[2]A. J. Bray and M. A. Moore, Phys. Rev. B **31**, 631 (1985); W. L. McMillan, Phys. Rev. B **31**, 340 (1985); W. L. McMillan, Phys. Rev. B **31**, 340 (1985); R. N. Bhatt and A. P. Young, Phys. Rev. Lett. **54**, 924 (1985); A. T. Ogielski and I. Morgenstern, Phys. Rev. Lett. **54**, 928 (1985).

[3]D. S. Fischer and D. A. Huse, Phys. Rev. Lett. **56**, 1601 (1986); J. Villain, Europhys. Lett. **2**, 871 (1986).

[4]R. J. Glauber, J. Math. Phys. (N.Y.) **4**, 294 (1963); M. Suzuki and R. Kubo, J. Phys. Soc. Jpn. **24**, 51 (1968).

[5]D. Sherrington and S. Kirkpatrick, Phys. Rev. Lett. **35**, 1792 (1975).

[6]J. Villain, Phys. Rev. Lett. **52**, 1543 (1984); D. S. Fisher, Phys. Rev. Lett. **56**, 416 (1986).

[7]H. Sompolinsky and A. Zippelius, Phys. Rev. Lett. **47**, 359 (1981), and Phys. Rev. B **25**, 6860 (1982).

[8]A. Zippelius, Phys. Rev. B **29**, 2717 (1984).

[9]H.-J. Sommers and K. H. Fischer, Z. Phys. B **58**, 125 (1985).

[10]W. Kinzel and K. H. Fischer, Solid State Commun. **23**, 682 (1977); S. Kirkpatrick and D. Sherrington, Phys. Rev. B **17**, 4384 (1978); K. H. Fischer, Solid State Commun. **46**, 309 (1983); A. Togashi and M. Suzuki, J. Phys. Soc. Jpn. **52**, 2994 (1983).

[11]H. Sompolinsky, Phys. Rev. Lett. **47**, 935 (1981); H.-J. Sommers, Z. Phys. B **50**, 97 (1983).

[12]J. R. L. de Almeida and E. J. S. Lage, J. Phys. C **16**, 939 (1983).

[13]H. J. Sommers and W. Dupont, J. Phys. C **17**, 5785 (1984).

[14]G. Parisi, Phys. Rev. Lett. **43**, 1754 (1979).

[15]D. J. Thouless, P. W. Anderson, and R. G. Palmer, Philos. Mag. **35**, 593 (1977).

[16]H. Horner, Z. Phys. B **57**, 29, 39 (1984).

[17]M. Mezard, G. Parisi, N. Sourlas, G. Toulouse, and M. Virasoro, Phys. Rev. Lett. **52**, 1156 (1984).

[18]C. De Dominicis and A. P. Young, J. Phys. A **16**, 2063 (1983).

[19]G. Parisi, Phys. Rev. Lett. **50**, 1946 (1983).

[20]H.-J. Sommers, unpublished. A more detailed version of this paper will contain proofs.

[21]M. Thomson, M. F. Thorpe, T. C. Choy, D. Sherrington, and H.-J. Sommers, Phys. Rev. B **33**, 1931 (1986).

[22]N. D. Mackenzie and A. P. Young, Phys. Rev. Lett. **49**, 301 (1982).

[23]H.-J. Sommers, J. Phys. A **16**, 447 (1983).

[24]J. A. Hertz, G. Grinstein, and S. A. Solla, to be published.