Off-Diagonal Long-Range Order, Oblique Confinement, and the Fractional Quantum Hall Effect

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We demonstrate the existence of a novel type of off-diagonal long-range order in the fractional-quantum-Hall-effect ground state. This is revealed for the case of fractional filling factor $v=1/m$ by application of Wilczek's "anyon" gauge transformation to attach m quantized flux tubes to each particle. The binding of the zeros of the wave function to the particles in the fractional quantum Hall effect is a (2+1)-dimensional analog of *oblique confinement* in which a condensation occurs, not of ordinary particles, but rather of composite objects consisting of particles and gauge flux tubes.

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A remarkable amount of progress has recently been made in our understanding of the fractional quantum Hall effect (FQHE)[1] following upon the seminal paper by Laughlin.[2] There remains, however, a major unsolved problem which centers on whether or not there exists an order parameter associated with some type of symmetry breaking.[3-6] The apparent symmetry breaking associated with the discrete degeneracy of the ground state in the Landau gauge[5] is an artifact of the toroidal geometry[6,7] and is not an issue here. Rather, the questions that we are addressing have been motivated by the analogies which have been observed to exist[4,8] between the FQHE and superfluidity and by recent progress towards a phenomenological Ginsburg-Landau picture of the FQHE.[4] Further motivation has come from the development of the correlated-ring-exchange theory of Kivelson *et al.*[9] (see also Chui, Hakim, and Ma,[10] and Chui,[10] and Baskaran[11]). The existence of infinitely large ring exchanges is a signal of broken gauge symmetry in superfluid helium[12] and is suggestive of something similar in the FQHE. However, the concept of ring exchanges on large length scales has not as yet been fully reconciled with Laughlin's (essentially exact[7]) variational wave functions which focus on the short-distance behavior of the two-particle correlation function. Furthermore it is clear that we cannot have an ordinary broken gauge symmetry since the particle density (which is conjugate to the phase) is ever more sharply defined as the length scale increases. The purpose of this Letter is to unify all these points of view by demonstrating the existence of a novel type of off-diagonal long-range order (ODLRO) in the FQHE ground state.

In second quantization the one-body density matrix is given by

$$\rho(z,z')=\sum_{m,n}\varphi_m^*(z)\varphi_n(z')\langle 0|c_n^\dagger c_m|0\rangle, \quad (1)$$

where $\varphi_n(z)$ is the nth lowest-Landau-level single-particle orbital[1] in the symmetric gauge, and z is a complex representation of the particle position vector in units of the magnetic length.[1] It is an unusual feature of this problem that there is a unique single-particle state for each angular momentum. Hence by making only the assumption that the ground state is isotropic and homogeneous we may deduce $\langle 0|c_n^\dagger c_m|0\rangle=v\delta_{nm}$, and thereby obtain the powerful identity:

$$\rho(z,z')=vg(z,z')=(v/2\pi)\exp(-\tfrac{1}{4}|z-z'|^2)\exp[\tfrac{1}{4}(z^*z'-zz'^*)], \quad (2)$$

where $g(z,z')$ is the ordinary single-particle Green's function.[13]

We see from (2) that the density matrix is short ranged with a characteristic scale given by the magnetic length, just as occurs in superconducting films in a magnetic field.[14] The same result can be obtained within first quantization via the expression

$$\rho(z,z')=\frac{N}{Z}\int d^2z_2\cdots d^2z_N\,\Psi^*(z,z_2,\ldots,z_N)\Psi(z',z_2,\ldots,z_N), \quad (3)$$

where Z is the norm of Ψ.

If the lowest Landau level has filling factor $v=1/m$ and the interaction is a short-ranged repulsion, then in the low-electron mass limit,[7] the *exact* ground-state wave function is given by Laughlin's expression:

$$\Psi(z_1,\ldots,z_N)=\prod_{i<j}(z_i-z_j)^m\exp\left(-\tfrac{1}{4}\sum_k|z_k|^2\right). \quad (4)$$

Laughlin's plasma analogy[2,15] proves that the ground state is a liquid of uniform density so that Eq. (2) is valid. The rapid phase oscillations of the integrand in (3) cause ρ to be short ranged. There is, nevertheless, a peculiar type of long-range order hidden in the ground state. For reasons which will become clear below, this order is revealed by considering the singular gauge field used in the study of "anyons"[16,17]:

$$\mathcal{A}_j(z_j) = \frac{\lambda\Phi_0}{2\pi}\sum_{i\neq j}\nabla_j \mathrm{Im}\ln(z_j - z_i), \quad (5)$$

where $\Phi_0 = hc/e$ is the quantum of flux and λ is a constant. The addition of this vector potential to the Hamiltonian is not a true gauge transformation since a flux tube is attached to each particle. If, however, $\lambda = m$ where m is an integer, the net effect is just a change in the phase of the wave function:

$$\Psi_{\mathrm{new}} = \exp\left(-im\sum_{i<j}\mathrm{Im}\ln(z_i - z_j)\right)\Psi_{\mathrm{old}}. \quad (6)$$

Application of (6) to the Laughlin wave function (4) yields

$$\Psi(z_1,\ldots,z_N) = \prod_{i<j}|z_i - z_j|^m \exp\left(-\tfrac{1}{4}\sum_k |z_k|^2\right), \quad (7)$$

which is purely real and is symmetric under particle exchange for both even and odd m. Hence we have the remarkable result that both fermion and boson systems map into bosons in this singular gauge.

Substituting (7) into (3) and using Laughlins's plasma analogy,[2,15] a little algebra shows that the singular-gauge density matrix $\tilde{\rho}$ can be expressed as

$$\tilde{\rho}(z,z') = (\upsilon/2\pi)\exp[-\beta\Delta f(z,z')]\,|z-z'|^{-m/2}, \quad (8)$$

where $\beta \equiv 2/m$, and $\Delta f(z,z')$ is the difference in free energy between two impurities of charge $m/2$ (located at z and z') and a single impurity of charge m (with arbitrary location). Because of complete screening of the impurities by the plasma, the free-energy difference $\Delta f(z,z')$ rapidly approaches a constant as $|z-z'| \to \infty$. This proves the existence of ODLRO[18] characterized by an exponent $\beta^{-1} = m/2$ equal to the plasma "temperature." For $m=1$ the asymptotic value of Δf can be found exactly: $\beta\Delta f_\infty = -0.03942$. For other values of m, $\beta\Delta f(z,z')$ can be estimated either by use of the ion-disk approximation[2,15] or the static (linear response) susceptibility of the (classical) plasma calculated from the known static structure factor[8] (see Fig. 1).

The rigorous and quantitative results we have obtained above are valid for the case of short-range repulsive interactions for which Laughlin's wave function is exact. We now wish to use these results for a qualitative examination of more general cases and to deepen our understanding of the ODLRO. We begin by noting that $\tilde{\rho}$ can be rewritten in the ordinary gauge as

$$\tilde{\rho}(z,z') = \frac{N}{Z}\int d^2z_2\cdots d^2z_N \exp\left(-i\frac{e}{\hbar c}\int_{\mathbf{z}}^{\mathbf{z}'} d\mathbf{r}\cdot\mathcal{A}_1\right)\Psi^*(z,z_2,\ldots,z_N)\Psi(z',z_2,\ldots,z_N), \quad (9)$$

where $\mathbf{z}$ and $\mathbf{z}'$ are vector representations of z and z'. The line integral in (9) is multiple valued but its exponential is single valued because the flux tubes are quantized. The additional phases introduced by the singular gauge transformation will cancel the phases in Ψ nearly everywhere, and produce ODLRO in $\tilde{\rho}$ if and only if the zeros of Ψ (which must necessarily be present because of the magnetic field[19]) are bound to the particles. Thus ODLRO in $\tilde{\rho}$ *always* signals a condensation of the zeros onto the particles (independent of whether or not the composite-particle occupation of the lowest momentum state diverges[18]). Because the gauge field $\mathcal{A}_1$ depends on the positions of *all* the particles, $\tilde{\rho}$ differs not just in the phase but in *magnitude* from ρ. This multiparticle object, which explicitly exhibits ODLRO, is very reminiscent of the topological order parameter in the XY model[20] and related gauge models[21,22] and is intimately connected with the frustrated XY model which arises in the correlated-ring-exchange theory.[9]

For short-range interactions, the zeros of Ψ are directly on the particles and the associated phase factors are exactly canceled by the gauge term in (9) [see Eq. (7)]. As the range of the interaction increases, $m-1$ of the zeros move away from the particles but remain nearby

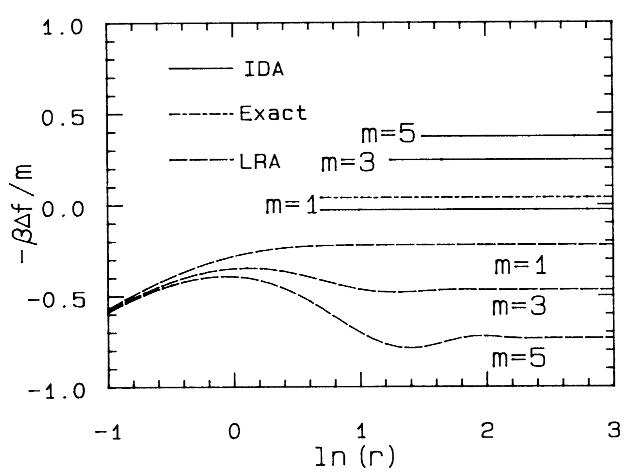

FIG. 1. Plot of $-\beta\Delta f(z,z')/m$ vs $r \equiv |z-z'|$ for filling factor $\upsilon = 1/m$. LRA is linear-response approximation, IDA is ion-disk approximation (shown only for separations exceeding the sum of the ion-disk radii). Because the plasma is strongly coupled, the IDA is quite accurate at $m=1$ and improves further with increasing m. The LRA is less accurate at $m=1$ and worsens with increasing m.

and bound to them.[7,23] The gauge and wave function phase factors in (9) now appear in the form of the bound vortex-antivortex pairs. We expect such bound pairs *not* to destroy the ODLRO and speculate (based on our understanding of the Kosterlitz-Thouless transition[20]) that the effect is at most to renormalize the exponent of the power law in (8). As the range of the potential is increased still further, numerical computations[7] indicate that a critical point is reached at which the gap rather suddenly collapses and the overlap between Laughlin's state and the true ground state drops quickly to zero. We propose that this gap collapse corresponds to the unbinding of the vortices and hence to the loss of ODLRO and the onset of short-range behavior of $\tilde{\rho}(z,z')$. Recall that the distinguishing feature of the FQHE state is its long wavelength excitation gap. At least within the single-mode approximation,[8] this gap can only exist when the ground state is homogeneous and the two-point correlation function exhibits perfect screening:

$$M_1 \equiv (\nu/2\pi)\int d^2r(r^2/2)[g(r)-1] = -1.$$

In the analog plasma problem, the zeros of Ψ act like point charges seen by each particle and the M_1 sum rule implies that electrons see each other as charge-$(m=1/\nu)$ objects; i.e., that m zeros are bound to each electron. Thus (within the single-mode approximation) there is a one-to-one correspondence between the existence of ODLRO and the occurrence of the FQHE.

The exact nature of the gap-collapse transition, which occurs when the range of the potential is increased,[7] is not understood at present. However, it has been proven[8] that the M_1 sum rule is satisfied by every homogeneous and isotropic state in the lowest Landau level. Hence the vortex unbinding should be a first-order transition to a state which breaks rotation symmetry (like the Tao-Thouless state[24]) and/or translation symmetry (like the Wigner crystal[4,8]). We know that as a function of *temperature* (for fixed interaction potential) there can be no Kosterlitz-Thouless transition[20] since isolated vortices (quasiparticles) cost only a finite energy in this system[4,25] (see, however, Ref. 10).

Further insight into the gap collapse can be obtained by considering the exceptional case of Laughlin's wave function with $m > 70$. In this case the zeros are still rigorously bound to the particles so that the analog plasma contains long-range forces (and $\tilde{\rho}$ exhibits ODLRO), but the plasma "temperature" has dropped below the freezing point.[2,15] If such a state exhibits (sufficiently[10]) long-range positional correlations, the FQHE would be destroyed by a gapless Goldstone mode associated with the broken translation symmetry. Hence in this exceptional case the normal connection between ODLRO and the FQHE would be broken by a gap collapse due to positional order at a finite wave vector.

The existence of ODLRO in $\tilde{\rho}$ is the type of infrared property which suggests that a field-theoretic approach to the FQHE would be viable. It is clear from the results presented here that the binding of the zeros of Ψ to the particles can be viewed as a condensation,[18] not of ordinary particles, but rather of composite objects consisting of a particle and m flux tubes. (We emphasize that these are *not* real flux tubes, but merely consequences of the singular gauge. The assumption that electrons can bind real flux tubes[26] is easily shown to be unphysical.[27]) The analog of this result for hierarchical daughter states of the Laughlin states[7,15] would be a condensation of composite objects consisting of n particles and m flux tubes (cf. Halperin's "pair" wave functions[19]). This seems closely analogous to the phenomenon of *oblique confinement*[22] and it ought to be possible to derive the appropriate field theory from first principles by use of this idea.

Since the singular gauge maps the problem onto interacting bosons, coherent-state path integration[28] may prove useful. A step in this direction has been taken recently in the form of a Landau-Ginsburg theory which was developed on phenomenological grounds.[4] In the static limit, the action has the "θ vacuum" form

$$S=\int d^2r\,|(-i\nabla+\mathbf{a})\psi(\mathbf{r})|^2+i\phi(\mathbf{r})(\psi^*\psi-1)-i(\theta/8\pi^2)(\phi\nabla\times\mathbf{a}+\mathbf{a}\times\nabla\phi), \tag{10}$$

where $\mathbf{a}$ is not the physical vector potential but an effective gauge field[4] representing frustration due to density deviations away from the quantized Laughlin density and ϕ is a scalar potential which couples both to the charge density and the "flux" density. From (10) the equation of motion for $\mathbf{a}$ is (in the static case):

$$\theta\nabla\times\mathbf{a}=(\psi^*\psi-1). \tag{11}$$

This equation and the parameter θ, which determines the charge carried by an isolated vortex, originally had to be chosen phenomenologically.[4] Now, however, it can be justified by examination of Eq. (5) which shows that the curl of $\mathcal{A}_j$ is proportional to the density of particles. If we identify $\mathbf{a}$ in (10) and (11) as

$$\mathbf{a}=\mathcal{A}_1+\mathbf{A}, \tag{12}$$

where $\mathbf{A}$ is the physical vector potential and we take $\psi^*\psi$ as the particle density relative to the density in the Laughlin state, then Eq. (11) follows from (5) with the θ angle being given by $\theta=2\pi/m$. This yields[4] the correct charge of an isolated vortex (Laughlin quasiparticle) of $q^*=1/m$. The connection between this result and the Berry phase argument of Arovas *et al.*[29] should be noted (see also Semenoff and Sodano[30]). To summarize, it is the strong phase fluctuations induced by the frustration associated with density variations [Eq. (11)] which pin the density at rational fractional values and account for the differences between the FQHE and ordinary superfluidity.[4]

We believe that these results shed considerable light on the FQHE, unify the different pictures of the effect, and emphasize the topological nature of the order in the zero-temperature state of the FQHE. The present picture leads to several predictions which can be tested by numerical computations by use of methods very similar to those now in use.[31] ODLRO will be found only in states exhibiting an excitation gap. The decay of the singular-gauge density matrix will be controlled by the distribution of distances of the zeros of the wave function from the particles. This distribution, which can be artificially varied by changing the model interaction, directly determines the short-range behavior of the density-density correlation function and hence the ground-state energy.[7,23]

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