# Phase Transitions in the Thermodynamic Formalism of Multifractals

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Nonanalyticities in the generalized dimensions of multifractal sets of physical interest are interpreted as phase transitions. The problem is mapped onto thermodynamics of one-dimensional spin models. The spin Hamiltonians are explicitly constructed and their phase transitions discussed. This mapping can provide insight in both directions.

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Multifractals are fractal sets which are not necessarily self-similar. Such sets appear in a variety of nonlinear physical phenomena like turbulence,[1] chaotic dynamical systems,[2] fractal growth processes,[3] etc., and have been also termed "fractal measures." To see the essential property which makes these sets nontrivial consider a set that can be covered at the $n$th level of construction by $a^n$ balls, where $(k-1)^n < a^n \leq k^n$, with $k$ integer. Assign to each ball an address $(\varepsilon_1, \ldots, \varepsilon_n)$ where $\varepsilon_i$ can take on $k$ values. Denote the radius of each ball by $l(\varepsilon_1, \ldots, \varepsilon_n)$. The crucial quantity that determines the properties of the set is the scaling function[4] $\sigma$, which is a daughter-to-mother ratio obtained in one refinement of the coverage:

$$\frac{l(\varepsilon_1, \ldots, \varepsilon_{n+1})}{l(\varepsilon, \ldots, \varepsilon_n)} = \sigma(\varepsilon_{n+1}, \ldots, \varepsilon_1). \tag{1}$$

Exactly self-similar sets have the property that $\sigma(\varepsilon_{n+1}, \ldots, \varepsilon_1) = \sigma(\varepsilon_{n+1})$. At each stage of the construction one splits the coverage as in the first stage. A spurt of theoretical (and some experimental[5]) activity started recently following the realization that many sets of interest are not self-similar, and in principle they have the property that $\sigma(\varepsilon_{n+1}, \ldots, \varepsilon_1)$ depends on the whole history of the construction.

It has been proposed that such sets can be usefully characterized by an infinite spectrum of generalized dimensions.[6] Defining the measure of a ball with address $(\varepsilon_1, \ldots, \varepsilon_n)$ by $p(\varepsilon_1, \ldots, \varepsilon_n)$, one introduces[7] the partition function $\Gamma(q,\tau) = \sum_{\varepsilon_1, \ldots, \varepsilon_n} p^q(\varepsilon_1, \ldots, \varepsilon_n)/l^\tau(\varepsilon_1, \ldots, \varepsilon_n)$. Next one argues that in the limit $n \to \infty$ the condition $\Gamma(q,\tau)=1$ singles out a quantity $\tau = \tau(q) = (q-1)D_q$, where $D_q$ are the generalized dimensions.[6] Appropriate Legendre transforms lead to the spectrum of singularities which was denoted $f(\alpha)$ in Ref. 7.

The aim of this Letter is to point out and discuss the fact that some sets of physical interest have a nonanalytic dependence of $D_q$ on $q$. Moreover, this phenomenon has a direct analogy to the phenomenon of phase transitions in condensed-matter physics. The fact that such a phenomenon exists was discovered first by Cvitanović in the context of the mode-locking structure of the circle map,[8] and later mentioned by Bohr and Rand,[9] Badii and Politi,[9] and Grassberger.[9] Here we put the analogy on a firm basis making full use of the recently developed thermodynamic formalism of multifractals.[10,11]

A quick path to this thermodynamic formalism[10,11] is opened by considering the special partitions that obey the conditions $p(\varepsilon_1, \ldots, \varepsilon_n) = \text{const} = a^{-n}$. Inserting this in the partition function and using the condition $\Gamma(q,\tau) = 1$ leads to the relation

$$a^{nq(\tau)} = \sum_{\varepsilon_1, \ldots, \varepsilon_n} |l(\varepsilon_1, \ldots, \varepsilon_n)|^{-\tau}, \tag{2}$$

where now $q(\tau)$ rather than $\tau(q)$ becomes the focus of analysis. We can now map the problem onto statistical mechanics of an $n$-dimensional spin system by dividing $a^{(n+1)q(\tau)}$ by $a^{nq(\tau)}$. With use of Eq. (2) we find (after adding summations on $\varepsilon_{nsspri}, \ldots, \varepsilon_{2sspri}$ which are compensated by Kronecker $\delta$'s)

$$\sum_{\substack{\varepsilon_1, \ldots, \varepsilon_{n+1}, \\ \varepsilon_2', \ldots, \varepsilon_n'}} \delta_{\varepsilon_n, \varepsilon_n'} \cdots \delta_{\varepsilon_2 \varepsilon_2'} \sigma^{-\tau}(\varepsilon_{n+1}, \ldots, \varepsilon_1) |l(\varepsilon_1, \varepsilon_2', \ldots, \varepsilon_n')| = a^{q(\tau)} \sum_{\varepsilon_1, \ldots, \varepsilon_n} |l(\varepsilon_1, \ldots, \varepsilon_n)|^{-\tau}. \tag{3}$$

Defining the transfer matrix

$$\langle \varepsilon_{n+1}, \ldots, \varepsilon_2 | T | \varepsilon_n', \ldots, \varepsilon_2', \varepsilon_1 \rangle = \sigma^{-\tau}(\varepsilon_{n+1}, \ldots, \varepsilon_1) \delta_{\varepsilon_2 \varepsilon_2'} \cdots \delta_{\varepsilon_n \varepsilon_n'},$$

we see that $a^{q(\tau)}$ is an eigenvalue of $T$. We thus have[12] a mapping on the thermodynamics and statistical mechanics of spin systems; $-\tau$ serves the role of the inverse temperature $\beta$; $-q(\tau)\ln(a)$ serves as the free energy $F(\beta)$. The thermodynamic system is a one-dimensional string of spin with a range of interaction that depends on the memory in $\sigma(\varepsilon_n, \ldots, \varepsilon_1)$. The number of spin states is $k$.

Before turning to the issue of phase transitions, we discuss the calculation of the Hamiltonian of the 1D spin system.

At the $n$th level of the construction we have the transfer matrix $T$ with elements $\sigma(\varepsilon_n, \ldots, \varepsilon_1)$ from which we can extract interactions involving $n$ spins. We write $E(\varepsilon_1, \ldots, \varepsilon_n) = \ln|\sigma(\varepsilon_n, \ldots, \varepsilon_1)|$, and

$$-E(\varepsilon_1, \ldots, \varepsilon_n) = \sum_{i=1}^{N} h_i^{(1)}\varepsilon_i + \sum_{i=1}^{n-1} k_i^{(2)}\varepsilon_i\varepsilon_{i+1} + \sum_{i=1}^{n-2} g_i^{(2)}\varepsilon_i\varepsilon_{i+2} + \cdots + m^{(n)}\varepsilon_1\cdots\varepsilon_n. \qquad (4)$$

At this point and in the rest of this Letter we shall think of binary $\varepsilon$'s which take on values $\pm 1$, and positive coefficients in (4) indicate ferromagnetic interactions. Naturally, raw calculations on given fractal sets yield interactions which are not translationally invariant, (i.e., $g_i^{(2)} \neq g_j^{(2)}$ for $i \neq j$ for example). However, we always use the freedom to seek translationally invariant interactions which lead to the same total Hamiltonian. This is obtained simply by our *summing up* all coefficients of the same type. The resulting translationally invariant interactions, i.e., $\tilde{h}^{(1)} = \sum_{i=1}^{n} h_i^{(1)}$, $\tilde{k}^{(2)} = \sum_{i=1}^{n-1} k_i^{(2)}$, etc., are the interactions that we quote below. The fact that the total Hamiltonian is the same can be seen easily.[12]

The two examples of strange sets that we consider here are (i) the Julia set[13] of the quadratic map $z' = g(z) = z^2 + 0.25$ with $z$ complex and (ii) the invariant measure of the map $x' = f(x) = 4x(1-x)$, $x \in [0,1]$. The second example is simpler since we have an explicit form for the invariant measure, $\rho(x) = \pi^{-1}[x(1-x)]^{-1/2}$. By the use of this, one can calculate explicitly $D_q$ and find[14]

$$D_q = \begin{cases} 1 \text{ for } q < 2, \\ q/[2(q-1)] \text{ for } q > 2. \end{cases} \qquad (5)$$

We thus have nonanalyticity at $q = 2$. The scaling function (1) can be obtained easily with the help of symbolic dynamics. Defining $\chi(x)$ to be $-1$ for $x < \frac{1}{2}$ and $+1$ for $x > \frac{1}{2}$ we seek intervals $l(\varepsilon_1, \ldots, \varepsilon_n)$ such that $\chi(x) = \varepsilon_1$, $\chi(f(x)) = \varepsilon_2, \ldots, \chi(f^{n-1}(x)) = \varepsilon_n$, for all $x$ in $l(\varepsilon_1, \ldots, \varepsilon_n)$. The measure of each such interval is precisely $2^{-n}$. The scaling function is then (1), and the result $\sigma(\varepsilon_n, \ldots, \varepsilon_1)$ for $n = 12$ is shown in Fig. 1. We note in passing that to our best knowledge this is the first scaling function obtained for a fully chaotic (nonhyperbolic) system, and that it converges very rapidly. We see that most of the values of $\sigma$ are clustered around $\frac{1}{2}$, but there are two boxes, namely $l(-1,-1,-1,\ldots)$ and $l(1,1,1, \ldots)$ that contribute a value $\frac{1}{4}$ to the scaling function. For $\tau$ positive and large these boxes dominate Eq. (2) and we have $2^{nq(\tau)} \sim \frac{1}{4}^{-n\tau}$ from which $q = 2\tau$. For $\tau$ negative the large boxes dominate and Eq. (2) reads $2^{nq(\tau)} \sim 2^n \frac{1}{2}^{-n\tau}$ or $q = 1 + \tau$. The transition is at $\tau = 1$ or $q = 2$ in agreement with (5). We note that in spin language the "states" $(-1,-1,-1,\ldots)$ and $(1,1,1,\ldots)$ are the "ground states" which are fully ordered. The only remaining puzzle is why the transition occurs at positive $\tau$, which in light of the identification $\tau \leftrightarrow -\beta$ indicates negative temperatures. We return to this point later.

The Julia-set example is much more subtle. This set can be obtained recursively from the preimages of the unstable fixed point which for $g(z) = z' + 0.25$ is exactly at $\bar{z} = \frac{1}{2}$. $|g'(\bar{z})|$ is therefore unity and the set is not hyperbolic. This is the reason for the phase transition (which does not occur in the hyperbolic Julia set of $z^2 + c$ for $-0.75 < c < 0.25$). The partition is again obtained with the help of symbolic dynamics. In the $n$th generation we consider[11] the $n$ symbols $\varepsilon_1, \ldots, \varepsilon_n$ obtained from writing $z(\varepsilon_1, \ldots, \varepsilon_n) = \pm[z'(\varepsilon_2, \ldots, \varepsilon_n) - \frac{1}{4}]^{1/2}$, where $\varepsilon_1 = -1$ when the positive branch is used and $\varepsilon_1 = 1$ for a negative branch. Writing then $t = \sum_{k=1}^{n}[(1+\varepsilon_k)/2]2^{-k}$ we denote $z(\varepsilon_1, \ldots, \varepsilon_n)$ by $z(t)$. The boxes, each of which has a measure $2^{-n}$, are defined by $l(\varepsilon_1, \ldots, \varepsilon_n) = |z(t + \frac{1}{2}n) - z(t)|$ and the scaling function follows.[11] The set as obtained at generation 12 is shown in Fig. 2(a) including the identification of $l(-1,-1,-1,\ldots)$. The scaling function for the same generation 12 is shown in Fig. 2(b). One of the problems is that the scaling functions converge excruciatingly slowly with the generations, but we expect that in the limit it reaches the value 1 for $X = 0, 1$ and the value 0 for $X = \frac{1}{2}$.

As a result of the fact that $|g'(\bar{z})| = 1$, the boxes $l(-1,-1,-1,\ldots)$ and $l(1,1,1,1,\ldots)$ decrease in size very slowly as the generation increases. In fact their size decreases like $1/n$ rather than exponentially, as almost all the other boxes do. Evidently for $\tau$ negative and large these boxes dominate, resulting in $q$ going to zero like $(\ln n)/n$. We thus expect $q(\tau \to -\infty)$ to be zero asymptotically. We also remember that $q = 0$ for $\tau = -D_0$ where $D_0$ is the Hausdorff dimension [remember that

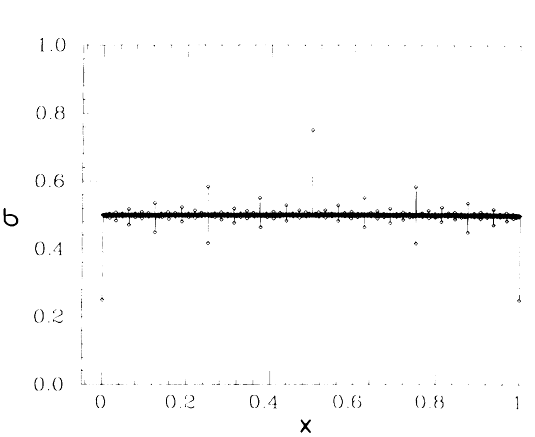


FIG. 1. Scaling function for the invariant measure of the map $x' = 4x(1-x)$. The abscissa is the number $x = \sum_{k=1}^{n}[(1+\varepsilon_k)/2]2^k/2^{n+1}$. Notice that for $x = 0, 1$, $\sigma = \frac{1}{4}$.

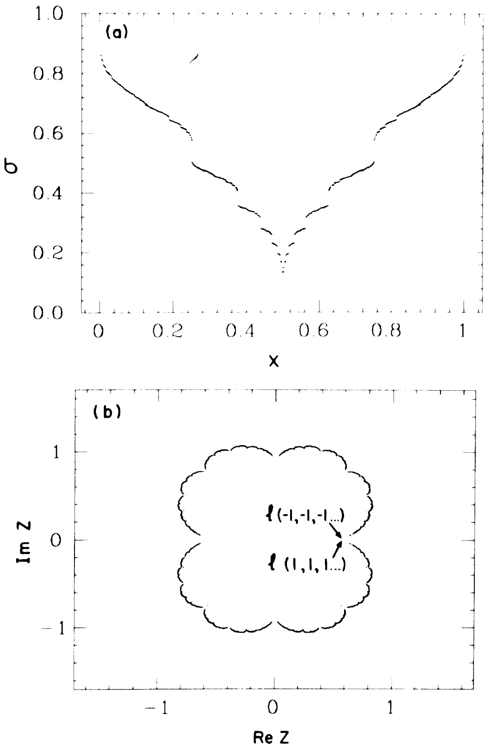


FIG. 2. (a) Scaling function for the Julia set as obtained for generation 12 from the map $z'=z^2+0.25$. The abscissa is the same as in Fig. 1. We expect that asymptotically $\sigma(0)=\sigma(1)=1$, and $\sigma(\frac{1}{2})=0$. (b) The Julia set of $z'=z^2+0.25$ at generation 12. $l(-1,-1,-1,\ldots)$ and $l(1,1,1,\ldots)$ are denoted.

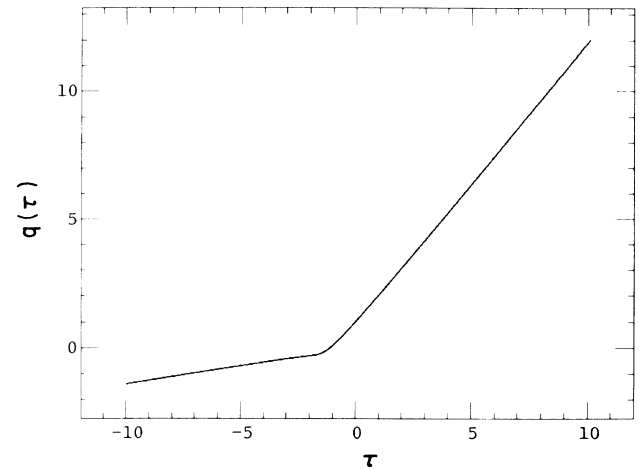


FIG. 3. Plot of $q(\tau)$ vs $\tau$ as obtained from the approximate data of 14 generations for the Julia set of $z'=z^2+0.25$. Notice that $q(\tau)$ is not yet zero for $\tau < -D_0$.

$\tau(q)=(q-1)D_q$]. Since from (2) it is clear that $q$ is monotonic in $\tau$ we conclude that $q$ is zero for all $\tau \leq -D_0$. On the other hand we know from the definition of the Hausdorff dimension that for $\tau > -D_0$ the right-hand side of Eq. (3) goes to infinity. Thus $q(\tau)$ cannot be zero for $\tau > -D_0$, and we conclude that at $\tau = -D_0$ the transition occurs. Figure 3 shows $q(\tau)$ as obtained numerically for generation $n=14$. The transition is clearly seen, but $q(\tau)$ is still nonzero for $\tau < -D_0$ because of the slow convergencelike $(\ln n)/n$.

The analytic properties near $\tau = -D_0$ are a delicate business. To argue that we have a true phase transition we turn now to the spin model. As explained above we evaluate the spin interactions from the scaling function. Because of the symmetry of the scaling function about $X=\frac{1}{2}$ there are no odd-spin interactions, i.e., no "magnetic field" and no three-spin interactions, etc. Next we examined the two-spin interactions for nearest neighbors, next-nearest neighbors, etc., up to a distance of 10 sites. These are all ferromagnetic, but are decreasing exponentially with the distance. As is well known one does not have a phase transition with only short-range two-spin interactions.[15] We have, however, multispin interactions as well, and they are all ferromagnetic. The situation is very close to the "droplet model" of Fisher,[16] and we use the results of that model to discuss the phase transition.

As argued by Fisher[6] one can consider the energy of a "cluster" of $m$ spins that all point, say, up. (In Fisher's model one thinks really about a liquid droplet, but the transition of spin language is immediate). One then writes the energy of the cluster as

$$E_m = -m\phi + W_m, \quad m \to \infty. \tag{6}$$

$\phi$ is the "bulk" contribution to the energy, whereas $W_m$ is the "surface" contribution. Fisher showed that if there are multispin interactions, then even though the two-spin interactions are short ranged, one can expect a transition at a finite temperature if $W_m/\ln m \to$ const. Moreover, this constant limit is $kT_c$ or $\beta_c^{-1}$. With the use of the computed Hamiltonian we calculated the energies of clusters of 2–12 spins, and found that Eq. (6) fits the numbers very well for $m \geq 4$, and a fit gives a value of $W_m/\ln m \simeq -1.44$. We thus expect a transition at $\tau \simeq -0.7$. In view of the fact that we use data from generation 12 to calculate the Hamiltonian, we find the agreement with $\tau = -D_0 \simeq -1.1$ to be reasonable. A similar test with the Julia set of $z^2+0.1$ resulted in a fit where $W_m/\ln m \to 0$. We thus conclude that at "low temperature", $(\tau < -D_0)$ the system is in a fully ordered state $(1,1,1,1,\ldots)$ or $(-1,-1,-1,\ldots)$ whereas at $\tau > -D_0$ it is disordered. On the basis of this analysis, we feel safe to conjecture that there is a true phase transition for this set, occurring when $-\tau$ equals the Hausdorff dimension.

A similar analysis has been performed on example (ii). Again, as a result of the symmetry of the scaling function, only even interactions appear. A major difference appears, however, when we compute the two-spin interactions of this problem. These are essentially independent of the distance. Moreover they are all antiferromagnetic! Thus a state $1, -1, 1, -1, \ldots$ is very frustrated and in fact cannot serve as a ground state. We thus understand why the transition appears at negative temperatures. Formally if we change $\beta \rightarrow -\beta$ and make antiferromagnetic interaction ferromagnetic, the problem remains invariant. The states $(1,1,1,\ldots)$ and $(-1,-1,-1,\ldots)$ become now ground states. Evidently, with two-spin interactions alone the system would have remained ordered at all (negative) temperatures. One has to examine carefully the multispin interactions and this is already beyond the scope of this Letter. These considerations, in addition to more details on example (i) and other cases, will appear elsewhere.

In summary, we have shown that an interesting phenomenon pertaining to multifractal sets can be mapped onto phase transitions of spin models, yielding valuable insight into the nature of such sets. We stress that the mapping can be used in the other direction as well, to gain insight on condensed-matter phenomena of interest.

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