Transition to Turbulence via Spatiotemporal Intermittency

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The transition to turbulence via spatiotemporal intermittency observed in a partial differential equation displays statistical features typical of critical phenomena. An analogy with directed percolation is drawn.

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The study of low-dimensional dissipative dynamical systems has provided a reasonable understanding of the process of transition to temporal chaos in strongly confined systems. The situation is much less advanced for weakly confined systems where chaos has both a spatial and temporal meaning as shown, for example, in convection. As is well known, these systems involve a large number of almost degenerate degrees of freedom. A conceptually simple way to increase this number consists of coupling identical dynamical systems. Choosing a coupling force proportional to the difference of state variables between neighbors, then at the continuous limit, one obtains models of the so-called reaction-diffusion type for which chaos should already present nontrivial spatiotemporal properties.[1] Staying at the discrete level, which is well in the spirit of the Poincaré mapping reduction procedure, arrays of coupled identical mappings have been studied.[2] However, it is not yet completely clear how far they can account for the continuous-space, continuous-time systems which they are supposed to model, and whether the kind of linear (quite passive) coupling used is a good (generic) representative of all spatial couplings and most especially of the nonlinear convective term essential to fluid dynamics. Thus, we were strongly motivated in studying a partial differential equation displaying a convective-type nonlinear term, steady cellular solutions as in convection, for example, and a transition to spatiotemporal chaos.

Here we consider the variant of the Swift-Hohenberg model of convection[3] currently named "model b" and studied mostly up to now from the point of view of wavelength selection close to the threshold.[4] This one-dimensional model reads

$$\partial_t W = \varepsilon W - (\partial_{xx}^2 + q_c^2)^2 W - W\partial_x W, \tag{1}$$

where ε is the control parameter. The trivial solution $W=0$ bifurcates at $\varepsilon=0$ towards a steady periodic solution. Contrary to the original model which derives from a potential and, as such, has only steady solutions, model b can also have periodic, quasiperiodic, and chaotic solutions for $\varepsilon<1$. $\varepsilon=1$ is a limiting value for which, up to a rescaling of W, x, and t, Eq. (1) is equivalent to the Kuramoto-Sivashinsky (KS) equation[5] in which the term $(1-\varepsilon)W$ can be understood as an additional damping $\eta\psi$[6] $[\eta=\frac{1}{4}(1-\varepsilon)]$:

$$\partial_t\psi + \eta\psi + \partial_{xx}^2\psi + \partial_{xxxx}^4\psi + 2\psi\partial_x\psi = 0. \tag{2}$$

Mathematical properties of the KS equation ($\eta=0$) have been studied thoroughly.[7] When $\eta=0$, a single control parameter remains, the length L of the interval at which boundary conditions are imposed to the function. It can be reduced to a low-dimensional effective dynamical system (inertial manifold[7]) the dimension of which grows linearly with L.[8] But there exists also windows in L where steady cellular solutions are stable[5] and which can be reached quite suddenly after extremely long transients for L not too large.[9] This last phenomenon seems more dramatic with periodic boundary conditions than with so-called "rigid boundary conditions,"[9a]

$$\psi(0)=\psi(L)=0,\quad \partial_x\psi(0)=\partial_x\psi(L)=0, \tag{3}$$

but bulk statistical properties of turbulent solutions at the limit of very large L do not seem sensitive to it.

Back to the damped KS equation, between the "convective threshold" (now at $\eta=\frac{1}{4}$ for L infinite) and the KS limit ($\eta=0$) enough space is left for a transition from steady rolls to "phase turbulence" as soon as L is large enough. At given L, this transition is controlled by η. We defer a complete report on the bifurcation diagram in the (L,η) plane and restrict ourselves to the large-L limit where confinement effects are weak, here $L>100$. At such values, solutions to the pure KS equation ($\eta=0$) are strongly chaotic (chaos first appears for $L\simeq15$). Moreover, the transition is rather reminiscent of crises typical of low-dimensional dynamical systems involving a small number of nonlinear steady solutions, some regular and the others with phase defects (strange steady states of Ref. 9c). Intermittency is not absent but has only a temporal meaning, either linked to these crises or to the fact that weakly unstable well-defined states can attract temporarily the system in a given region of the inertial manifold. The transition to weak turbulence looks quite different for the damped KS equation at large L. In the transition region, weak turbulence takes the form of a fluctuating mixture of regular and turbulent domains with well-defined boundaries. Such a regime is often called "spatiotemporal intermittency."[2] The transition is continuous in the sense that (i) turbulent domains slowly invade the system above some threshold and slowly recede below it and (ii) global statistical properties evolve gently in this parameter range.

Simulations on model (3) with rigid boundary conditions (4) have been performed with the use of a standard finite-difference numerical code, second order in space and time (Crank-Nicolson Adams-Bashford). Care has been taken to vary the spatiotemporal resolution in order to check the reliability of the phenomena reported. We used typically $\delta x = 0.25$ (17 points per roll) and $\delta t = 0.1$. Values of L varied from 200 to 3200. $L = 800$ seems to be the lower bound above which the transition to chaos takes place through spatiotemporal intermittency.

At moderate aspect ratio ($L = 100$), a subcritical Hopf bifurcation towards an oscillatory state takes place followed by a supercritical transition to a quasiperiodic state.[10a] Time series used to detect these regimes as well as other tools popularized by the study of dissipative dynamical systems (Lyapunov exponents...) have turned out to be of little help for the elucidation of these spatial features. On the other hand, simply plotting the zeros or the extrema of the field variable already reveals them.[10b] The origin of the first frequency can be traced back to an instability involving the basic wave vector selected by boundary effects, its $\frac{1}{2}$ subharmonic (spatial period doubling) and their harmonics ($\frac{3}{2}$ and 2 in particular). The second frequency is related to the regular propagation of a phase disturbance. The propagation velocity v has been found roughly independent of L for $L > 100$ and in the narrow range of η values concerned ($v = 0.8$). No detailed theoretical account of these phenomena has been given up to now, but it is likely that some amplitude-equation formalism would be adapted,[11] provided that end-effect perturbations be included.

In the range $100 < L < 800$, the transition to turbulence takes place through a loss of spatial coherence of the propagating waves at the origin of quasiperiodicity.[10] Two processes have been observed to occur, a nucleation of defects, essentially space-time dislocations[9b] in the bulk, and a damping of the waves as they penetrate in regular domains referred to as "coherent structures." We have also observed a multiplicity of asymptotic states, some of which were regular, quasiperiodic, but with weaker and weaker oscillations in the middle, subsequent regimes being extremely weakly chaotic with slightly irregular oscillations confined to boundary layers. Others already presented a precursor of spatiotemporal intermittency to be described below. This multiplicity seemed related to the persistence of sizable end effects, which become less and less pronounced as L increases.

At large aspect ratio, a further compression of the spatiotemporal information is crucially needed: The underlying periodicity of the roll structure must be rubbed out in order to focus the attention on the distortions. It turns out that in steady regions (called laminar in the present context) the local peak-to-peak amplitude of the rolls is larger than in turbulent regions. Thus, setting a cutoff, one can easily identify turbulent and laminar domains. We have checked that the qualitative features of the picture obtained in that way were insensitive to the precise value of the cutoff. Figure 1 displays such an all-or-nothing representation for $L = 1600$ and $\eta = 0.080$ on a time interval of length 5000. Dark regions are laminar while turbulent patches are left blank. Propagating coherent oscillations particularly visible in one boundary layer but also decorating the frontier between turbulent and laminar domains come and complicate slightly the picture.

Figure 1 can be analyzed as a juxtaposition of very

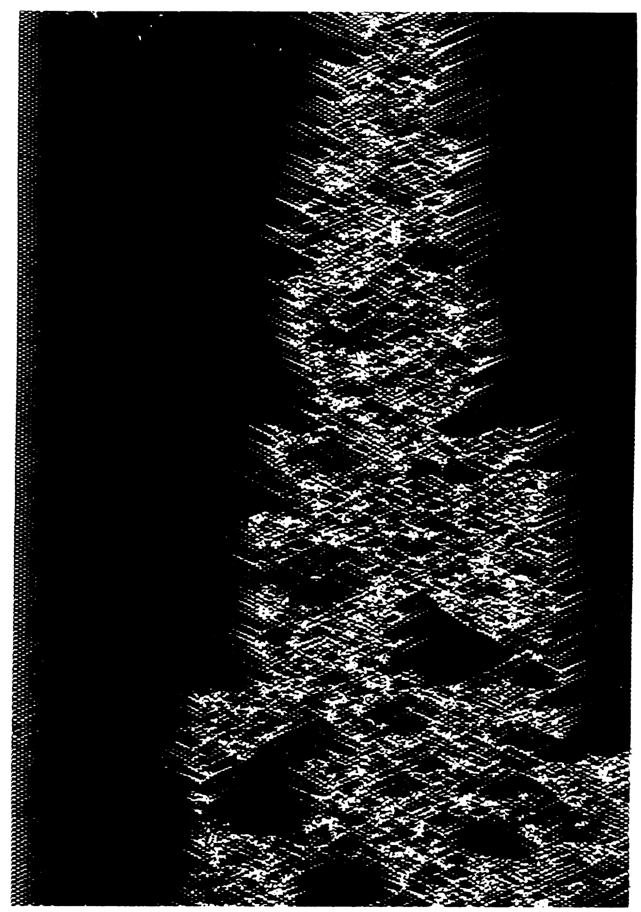

FIG. 1. Solution of model (3) for $L = 1600$ and $\eta = 0.080$. Typical example of the death of a spatiotemporally intermittent domain, receding at a well-defined velocity, close to the threshold for sustained intermittency (time is running upwards).

large laminar domains close to the boundaries and a "mixed state" made of interwoven turbulent and laminar domains of smaller size. Large laminar domains with typical sizes of the order of L correspond to spatially truncated asymptotic states of nearly steady roll patterns seen closer to the convective threshold. The boundary of the mixed state is well defined and its velocity can be measured. For $L=1600$ and $\eta=0.080$ the mixed state is seen to recede regularly at a velocity $V=0.09$. Around $\eta=0.078$ the situation becomes quite confused, certain initial conditions lead to a nearly steady sharing of the system between laminar and mixed state, i.e., $V=0$, others to completely laminar or, on the contrary, completely chaotic states, thus testifying for still sizable end effects. For $\eta=0.070$, no coherent structure at a scale of the order of L can be found any longer; moreover, the mixed state invades the system at a velocity $V=0.15$ so that the system is already deep in a new regime of sustained spatiotemporal intermittency. Thus we can safely say that a change of behavior takes place at about $\eta=0.078$ for $L=1600$. In the range $L=800-3200$ the value of the intermittency threshold seems roughly independent of size effects. The invasion or recession velocity V is much smaller than the propagation velocity v or a phase disturbance which, as well as the underlying roll wavelength or the attenuation length of oscillations in locally steady rolls, plays the role of a microscopic parameter as opposed to the length of laminar or turbulent regions at given η.

The transition to turbulence via spatiotemporal intermittency thus presents itself as a continuous process much reminiscent of a second-order phase transition. The study of the distribution of lengths of laminar domains confirms the existence of a critical region. This is clearly shown in Fig. 2 which displays the histograms corresponding to $L=1600$ for $\eta=0.078$ and $\eta=0.040$. They give the statistics, cumulated on a time interval of 20000, of the number of laminar domains with a given length. At threshold, one gets a power-law behavior with a characteristic exponent τ of the order of 3.15, while far from the threshold the decay is exponential with a characteristic length ξ of the order of 3.5. No power law can be extracted for $\eta=0.070$, which suggests that the system is operating at some crossover regime.

If one follows a suggestion of Pomeau,[12] it is very tempting to interpret the transition to turbulence via spatiotemporal intermittency described above as a directed-percolation process.[13] This analogy would account for the existence of a well-defined threshold, scaling in the critical region, the opening of the observation angle, etc. The introduction of probabilities in a deterministic problem seems necessary to avoid a spurious sensitivity of the coherence of chaotic states to the coherence of initial conditions which characterizes deterministic automata.[14] In addition, this introduction would be some sort of short circuit, allowing for the instability and subsequent stochasticity of localized degrees of freedom,[15] therefore jumping directly over the intermediate step of discrete space-time systems with deterministic mappings close to some temporal intermittency threshold.[2] Regarding this directed-percolation type of approach much remains to be done. From an experimental point of view, a better characterization of the possible critical behavior of the front velocity and a statistics going beyond space coherence and accounting also for time coherence are already in progress, aiming at the determination of new critical exponents. From a theoretical point of view, the main challenge would be the understanding of the relation between the transfer of a dynamical information (instantaneous state of a given localized degree of freedom) and

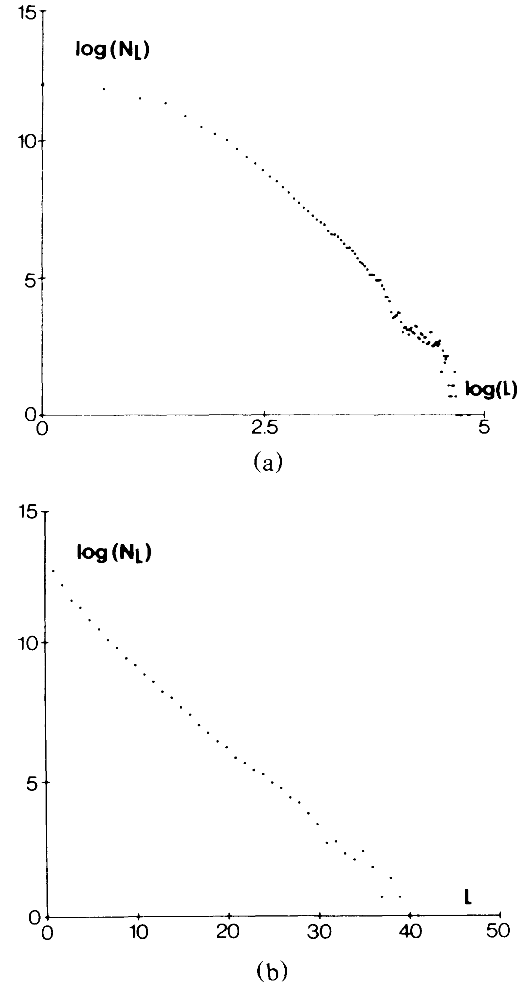

FIG. 2. Histograms of the lengths of laminar domains in the mixed state (in units of the half of the most unstable wavelength). (a) Close to the threshold ($\eta=0.078$), log-log plot, algebraic decay with exponent $\tau=3.15$ and (b) deep in the turbulent regime ($\eta=0.040$), semilogarithmic plot, exponential decay with a characteristic length $\xi=3.5$.

its translation as a topological information (part of a laminar or turbulent domain), which, according to Pomeau, could account for quantitative discrepancies with the strictly probabilistic problem.

Another challenge is the determination of the degree of universality of spatiotemporal intermittency displayed by the model studied here. This behavior is obviously connected to that observed in lattices of coupled mappings.[2] A novel interpretation of the data obtained with such models would be profitable, particularly with respect to the current belief that only a small number of degrees of freedom is involved, linked to the dynamics of a small number of kinks or defects[2b] or more generally of nonlinear eigenmodes on some low-dimensional manifold,[7] a statistical-mechanics-type approach[9b] could turn out to be more appropriate.

In any case, spatiotemporal intermittency is ubiquitous in hydrodynamic turbulence. Phenomenological descriptions of the transition process in pipe flows or boundary-layer flows, etc.,[16] seem sufficiently similar to that obtained here to motivate a continued effort along the direction of our current work.

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