Polarization Solitons

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We consider the interaction of counterpropagating pulses via a $\chi^{(3)}$ nonlinearity, in an isotropic electrostriction medium. The polarization is taken into account and can vary through each pulse; group-velocity dispersion is neglected. We find a family of analytic solutions, termed polarization solitons, which play a central role in the dynamics. Upon interaction, the net effect on their polarization profiles is a rigid rotation in Stokes-vector space. As special cases, they include the first nontrivial examples of polarization-transparent and orthogonally switched pulses.

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Solitons and solitary waves[1] in nonlinear optical media have received a great deal of attention in recent years both as an intriguing topic in mathematical physics, and because of interest in their application to optical devices. They arise for example with self-modulated pulses in the presence of group-velocity dispersion[2] and in certain multiwave-mixing processes.[3] In this Letter we consider a degenerate four-wave interaction consisting of two counterpropagating pulses in a $\chi^{(3)}$ medium, with arbitrary polarizations which can vary through each pulse. We neglect group-velocity dispersion, which results in the intensity profiles propagating unchanged from their initial forms. The polarization, however, evolves according to the nonlinear interaction and it is found that it can exhibit soliton behavior.

We have discussed some general results for the *steady-state* spatial polarization distribution of counterpropagating plane waves, for all propagation-axis rotation symmetries C_n, $n \geq 1$, in parity-invariant and -noninvariant media.[4,5] Previous workers considered the more restricted cases of isotropic media[6,7] and a C_4 propagation axis[8] in parity-invariant media. The analogous dynamical problem is significantly more difficult; we consider the simplest case of an isotropic parity-invariant medium with the electrostriction mechanism of the nonlinearity.[9] Furthermore, the response time of the nonlinearity is assumed to be instantaneous

In such a medium, the third-order complex polarization can be written[10]

$$\mathbf{P}^{(3)} = A(\mathbf{E}^* \cdot \mathbf{E})\mathbf{E}, \tag{1}$$

where for a lossless medium A is real. We introduce the slowly varying amplitudes $\mathbf{e}_1$ and $\mathbf{e}_2$ to describe pulses propagating in the $+\hat{\mathbf{z}}$ and $-\hat{\mathbf{z}}$ directions, respectively. For a small nonlinearity, in the plane-wave approximation we deduce in the usual manner the equations of motion,

$$\begin{aligned}(\partial/\partial\tau + \partial/\partial\xi)e_{1i} &= i\lambda\{[(\mathbf{e}_1^* \cdot e_1) + (\mathbf{e}_2^* \cdot \mathbf{e}_2)]e_{1i} + (\mathbf{e}_2^* \cdot \mathbf{e}_1)e_{2i}\}, \\ (\partial/\partial\tau - \partial/\partial\xi)e_{2i} &= i\lambda\{[(\mathbf{e}_2^* \cdot \mathbf{e}_2) + (\mathbf{e}_1^* \cdot \mathbf{e}_1)]e_{2i} + (\mathbf{e}_1^* \cdot \mathbf{e}_2)e_{1i}\},\end{aligned} \tag{2}$$

where the index i runs over the two transverse components (x and y, which we denote by 1 and 2). Here τ and ξ are dimensionless time and space coordinates, and λ characterizes the strength of the nonlinearity,

$$\tau \equiv k_0\omega_0' t, \quad \xi \equiv k_0 z, \quad \lambda \equiv 2\pi A/\varepsilon_0. \tag{3}$$

Further, k_0 is the carrier wave number, ω_0' is the group velocity (at the carrier frequency), and $\varepsilon_0 \equiv 1 + 4\pi\chi(\omega_0)$, where χ is the isotropic linear susceptibility. In arriving at Eqs. (2) we have neglected group-velocity dispersion.

This problem is best approached from a Stokes-vector formalism, which clearly elucidates the polarization aspects. We define the Stokes vectors[11]

$$s_i \equiv e_{1j}^*(\sigma)_{jk} e_{1k}, \quad t_i \equiv e_{2j}^*(\sigma)_{jk} e_{2k}, \quad i = 1,2,3, \tag{4}$$

where σ_i are the Pauli spin matrices, and the indices j and k are summed over 1 and 2. The magnitude of each Stokes vector is simply proportional to the respective intensities,

$$s_0 \equiv |\mathbf{s}| = \mathbf{e}_1^* \cdot \mathbf{e}_1, \quad t_0 \equiv |\mathbf{t}| = \mathbf{e}_2^* \cdot \mathbf{e}_2. \tag{5}$$

The correspondence between the direction of the Stokes vector and the polarization is indicated by the Poincaré sphere (see Fig. 1 and, e.g., Ref. 4). We also make a change of independent variables to

$$x \equiv \tfrac{1}{2}(\xi - \tau), \quad y \equiv -\tfrac{1}{2}(\xi + \tau), \tag{6}$$

which represent spatial coordinates in the moving reference frame of the first and second pulse, respectively. The leading and trailing edges of the pulses are described by $(x,y) \to \infty$ and $(x,y) \to -\infty$, respectively.

Using Eqs. (2), (3), (4), and (6) we deduce the Stokes-vector equations of motion,

$$\partial\mathbf{s}/\partial y = \lambda\mathbf{t}\times\mathbf{s}, \quad \partial\mathbf{t}/\partial x = \lambda\mathbf{s}\times\mathbf{t}. \tag{7}$$

These are coupled Bloch equations which describe precession of the Stokes vectors about one another. In the general case of arbitrary initial polarizations, which can vary through each pulse, this motion is nontrivial. Even in the case of initially constant polarizations the dynamics is complicated since the leading edge of each pulse distorts the polarization with which the remainder of the pulse interacts. However, if we take the dot products of

s and **t** with their respective equations of motion, we find

$$s_0(x,y)=s_0(x), \quad t_0(x,y)=t_0(y), \tag{8}$$

and thus the intensity profiles counterpropagate, at the group velocity, unchanged from their initial forms. This is a direct consequence of our neglect of group-velocity dispersion: The intensity profiles are fixed by initial conditions. We shall consider these profiles as arbitrary, assuming only that they are of finite area. While the pulses are widely separated, the polarization configurations also propagate unchanged in form. Of course while the intensity profiles are overlapping the polarizations evolve according to Eqs. (7).

Uniform polarization solutions of Eqs. (7) ($\hat{\mathbf{s}}=\pm\hat{\mathbf{t}}$) have been discussed in the context of plane waves.[5,7] The limit where one pulse intensity is very small has also been considered, in an electric field formalism.[12] We can considerably extend these results to the nontrivial case when both intensities are of significant magnitude. Returning to Eqs. (7), upon addition we find

$$\partial\mathbf{s}/\partial y+\partial\mathbf{t}/\partial x=0, \tag{9}$$

which suggests that we introduce a vector potential $\mathbf{v}$ such that

$$\mathbf{s}=\partial\mathbf{v}/\partial x, \quad \mathbf{t}=-\partial\mathbf{v}/\partial y. \tag{10}$$

Then Eq. (9) is automatically satisfied and from either of relations (7) we obtain the governing equations for the vector potential,

$$\begin{aligned}
\frac{\partial^2 v_1}{\partial x\partial y}&=\lambda\left[\frac{\partial v_2}{\partial x}\frac{\partial v_3}{\partial y}-\frac{\partial v_2}{\partial y}\frac{\partial v_3}{\partial x}\right],\\
\frac{\partial^2 v_2}{\partial x\partial y}&=\lambda\left[\frac{\partial v_3}{\partial x}\frac{\partial v_1}{\partial y}-\frac{\partial v_3}{\partial y}\frac{\partial v_1}{\partial x}\right],\\
\frac{\partial^2 v_3}{\partial x\partial y}&=\lambda\left[\frac{\partial v_1}{\partial x}\frac{\partial v_2}{\partial y}-\frac{\partial v_1}{\partial y}\frac{\partial v_2}{\partial x}\right].
\end{aligned} \tag{11}$$

Though these equations are second order they involve only three unknown functions $\mathbf{v}$, as compared to six in Eqs. (7), and the right-hand sides are simply Jacobian determinants. We shall obtain a family of analytic solutions by restricting our search to functions for which one of the Jacobians vanishes identically, let us say the second one:

$$\frac{\partial(v_3,v_1)}{\partial(x,y)}=0. \tag{12}$$

The equation of motion for v_2 can then be integrated to obtain

$$v_2(x,y)=a(x)+b(y), \tag{13}$$

where $a(x)$ and $b(y)$ are arbitrary functions, which we will eventually relate to initial conditions. Because of some basic properties of Jacobian determinants,[13] Eq. (12) allows us to write v_3 as a function of v_1, or vice versa. Using this property, and restricting ourselves further to functions v_1 of the form

$$v_1(x,y)=v_1(c(x)+d(y)), \tag{14}$$

where $c(x)$ and $d(y)$ are unknown functions to be determined, we are able to obtain the solution

$$\begin{aligned}
v_1(x,y)&=-\frac{\sin\theta_s\sin\theta_t}{\lambda\sin(\theta_t-\theta_s)}\cos[\phi_s(x)\pm\phi_t(y)+\phi_0],\\
v_2(x,y)&=\frac{\sin\theta_t\cos\theta_s\phi_s(x)\pm\sin\theta_s\cos\theta_t\phi_t(y)}{\lambda\sin(\theta_t-\theta_s)},\\
v_3(x,y)&=\frac{\sin\theta_s\sin\theta_t}{\lambda\sin(\theta_t-\theta_s)}\sin[\phi_s(x)\pm\phi_t(y)+\phi_0].
\end{aligned} \tag{15}$$

Here θ_s, θ_t, and ϕ_0 are arbitrary constants ($\theta_s\neq\theta_t$, $0<\theta_s,\theta_t<\pi$, $0\le\phi_0\le 2\pi$) specified by initial conditions, and

$$\begin{aligned}
\phi_s(x)&=\lambda\frac{\sin(\theta_s-\theta_t)}{\sin\theta_t}\int_x^\infty s_0(x')dx',\\
\phi_t(y)&=\lambda\frac{\sin(\theta_t-\theta_s)}{\sin\theta_s}\int_y^\infty t_0(y')dy',
\end{aligned} \tag{16}$$

where $s_0(x)$ and $t_0(y)$ are also specified by initial conditions. Then from Eqs. (10) and (15) we obtain our solutions in Stokes-vector form,

$$\begin{aligned}
s_1(x,y)&=s_0(x)\sin\theta_s\sin[\phi_s(x)\pm\phi_t(y)+\phi_0],\\
s_2(x,y)&=s_0(x)\cos\theta_s,\\
s_3(x,y)&=s_0(x)\sin\theta_s\cos[\phi_s(x)\pm\phi_t(y)+\phi_0],\\
t_1(x,y)&=\pm t_0(y)\sin\theta_t\sin[\phi_s(x)\pm\phi_t(y)+\phi_0],\\
t_2(x,y)&=\pm t_0(y)\cos\theta_t,\\
t_3(x,y)&=\pm t_0(y)\sin\theta_t\cos[\phi_s(x)\pm\phi_t(y)+\phi_0].
\end{aligned} \tag{17}$$

Substitution of these expressions directly into Eqs. (7) verifies that they are indeed a solution. These are also well defined for $\theta_s=\theta_t$, in which case they reduce to the trivial steady-state solutions $\mathbf{s}=\pm\hat{\mathbf{t}}$, mentioned previously. The angles θ and ϕ are identified as spherical polar coordinates in Stokes-vector space with the 2 axis as the polar axis.

To understand the nature of the solutions (17), we look first at the polarizations of the pulses when they are widely separated. The initial polarization distribution of the s pulse is described by Eqs. (17) upon our setting $y=\infty$, which in turn implies $\phi_t(y)=0$. Moving through the pulse (x ranging from $+\infty$ at the leading edge to $-\infty$ at the trailing edge) the Stokes vector $\mathbf{s}$ describing the local polarization is at a fixed angle θ_s from the 2 axis in Stokes-vector space (see Fig. 1). But it rotates about that axis, beginning at an angle ϕ_0 from the 3 axis at the leading edge of the pulse; moving towards the trailing edge of the pulse, the rotation at a given point is

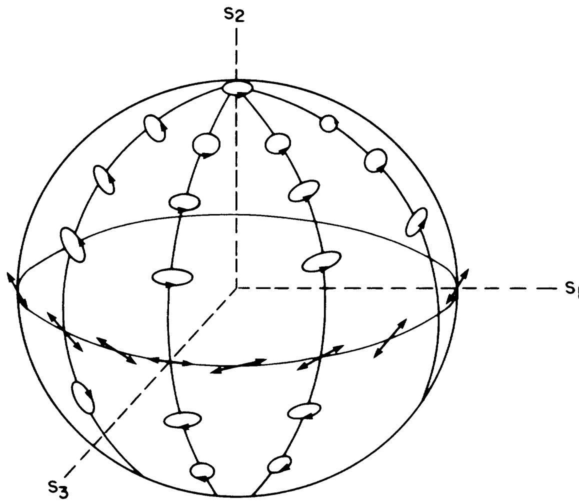

FIG. 1. The Poincaré sphere showing the correspondence between polarization and the directions of the Stokes vector.

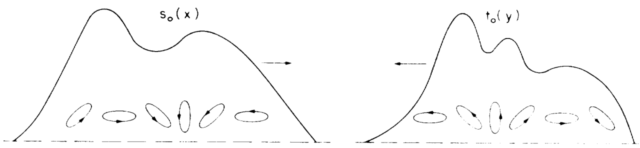

FIG. 2. Profile of the solitons whose polarization varies in a definite manner as a function of position in the pulse.

proportional to the area of the intensity in the pulse up to that point. That is, the ellipticity of the polarization is constant, but the angle that the major axis of the ellipse makes with a space-fixed direction varies in a definite way with the intensity distribution. A similar description applies to the initial distribution of polarization in the t pulse (see Fig. 2).

As the pulses interact their polarization distributions rotate about the polar axis in Stokes-vector space. The rotation angle at a given point in one pulse is proportional to the area, up to that point, of the other pulse; see Eqs. (16) and (17). Upon emerging from the interaction the net result is only a rigid rotation of the initial polarization distribution about the polar axis in Stokes-vector space. We refer to such solutions as "polarization solitons."

The emerging s pulse, obtained from Eqs. (16) and (17) by our setting $y=-\infty$, is

$$\begin{aligned} s_1(x) &= s_0(x)\sin\theta_s \sin[\phi_s(x)\pm\phi_t+\phi_0],\\ s_2(x) &= s_0(x)\cos\theta_s, \\ s_3(x) &= s_0(x)\sin\theta_s\cos[\phi_s(x)\pm\phi_t+\phi_0], \end{aligned} \tag{18}$$

where

$$\phi_t = \lambda\frac{\sin(\theta_t-\theta_s)}{\sin\theta_s}\int_{-\infty}^{\infty} t_0(y')dy'. \tag{19}$$

The rotation angle ϕ_t is simply proportional to the total area of the t pulse. If this angle is an integer multiple of 2π then the s pulse emerges from the interaction with its initial polarization configuration. An analogous argument applies to the emerging t pulse, and if the corresponding rotation angle is also a multiple of 2π then we have solutions which represent transparent pulses, as concerns the polarization. On the other hand if $\theta_s=\pi/2$ and the rotation angle ϕ_t is an odd-integer multiple of π then the emerging s pulse is orthogonally switched from its initial form, $\mathbf{s}(x)\to-\mathbf{s}(x)$.

At this point we note that the Stokes-vector equations of motion (7), as well as the vector-potential equations (11), are invariant under an arbitrary SO(3) rotation in Stokes-vector space. Thus an arbitrary SO(3) rotation of the solution (17) is also a solution. In other words, we can choose the polar axis in any direction in Stokes-vector space and we obtain a solution analogous to Eqs. (17). Thus the previous discussion applies to a whole family of solutions.

Finally, we consider the absolute phases of the electric fields, which are not described by the Stokes vectors. We write the slowly varying electric field amplitudes as

$$\mathbf{e}(x,y)=\{\hat{\mathbf{x}}\,|e_x(x,y)|\exp[i\Delta(x,y)]+\hat{\mathbf{y}}\,|e_y(x,y)|\exp[-i\Delta(x,y)]\}\exp[i\psi(x,y)], \tag{20}$$

where Δ describes the relative phase and ψ the absolute phase of the field. Within our approximations we obtain a closed systems of equations (7) for the Stokes vectors, that is, for the amplitudes and relative phases. The solutions that we found can be substituted back into the electric field equations of motion (2) to determine the absolute phases. In this manner, we obtain

$$\begin{aligned} \psi_s(x,y) &= \psi_s(x,y_0)-\lambda s_0(x)(y-y_0)-\frac{1}{2}\lambda\left(3\pm\frac{\sin\theta_t}{\sin\theta_s}\right)\int_{y_0}^{y} t_0(y')dy',\\ \psi_t(x,y) &= \psi_t(x_0,y)-\lambda t_0(y)(x-x_0)-\frac{1}{2}\lambda\left(3\pm\frac{\sin\theta_s}{\sin\theta_t}\right)\int_{x_0}^{x} s_0(x')dx' \end{aligned} \tag{21}$$

for the s and t pulses, where x_0 and y_0 denote initial coordinates. For pulses initially widely separated, x_0, $y_0 \gg 1$, the first term in each of Eqs. (21) is the initial phase variation. The second term in each describes self-phase modulation: an intensity-dependent wave vector and frequency shift. The last terms represent cross-phase modulations, which vanish initially and approach a constant as the pulsed become widely separated.

In conclusion, we have found a family of solutions—polarization solitons—which play a central role in the polarization dynamics of counterpropagating pulses. Until interaction their polarizations remain constant, and upon interaction the net effect is a rigid rotation in Stokes-vector space. These include the first nontrivial example of pulses which are transparent or orthogonally switched with respect to polarization. More general initial conditions and an extension of these results to arbitrary physical mechanisms of the nonlinearity is currently under study. Such pulses will probably constitute a better way to study such nonlinear systems than the steady-state solutions previously discussed,[5,7] since in practice the initial conditions can be more easily set.

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