Lévy Dynamics of Enhanced Diffusion: Application to Turbulence

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We introduce a stochastic process called a Lévy walk which is a random walk with a nonlocal memory coupled in space and in time in a scaling fashion. Lévy walks result in enhanced diffusion, i.e., diffusion that grows as t^α, $\alpha > 1$. When applied to the description of a passive scalar diffusing in a fluctuating fluid flow the model generalizes Taylor's correlated-walk approach. It yields Richardson's t^3 law for the turbulent diffusion of a passive scalar in a Kolmogorov $-\frac{5}{3}$ homogeneous turbulent flow and also gives the deviations from the $\frac{5}{3}$ exponent resulting from Mandelbrot's intermittency. The model can be extended to studies of chemical reactions in turbulent flow.

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In the 1920s and 1930s Lévy[1] was concerned with the question of when a sum of identically distributed random variables has the same probability distribution as any one of the terms in the sum. This is a question of scaling and the paradigm of fractals, i.e., when can a part have the same properties as the whole. Lévy completely solved this problem and the resultant distributions are now called Lévy's stable laws. In general, Lévy's laws deal with probability densities which have infinite moments and thus do not possess a definite scale as would be appropriate to Gaussian statistics, for example. While Lévy's has become a well-established field of mathematics, the appearance of infinite moments has blocked its usefulness in physical applications despite the 1954 prediction of Gnendenko and Kolmogorov[2] that "It is probable that the scope of applied problems in which they play an essential role will become in due course rather wide."

In this Letter we discuss the way in which Lévy distributions can be applied to the statistical-mechanical description of the dynamics of complex physical systems possessing many scales, and show how these distributions may be used to describe the phenomenon of *enhanced diffusion.* Sublinear diffusive growth in which the diffusion coefficient increases with time as t^α with $\alpha \le 1$ is familiar from disordered materials and trapping phenomenon in condensed matter physics. Enhanced diffusion with $\alpha > 1$ arises in such cases as phase diffusion in the chaotic regime of a Josephson junction,[3,4] chaos-induced turbulent diffusion,[5] the relation between the root-mean-square characteristic length of a polymer and the number of monomer units,[6] diffusion of a Brownian particle in a pure shear flow,[7] and the zero-component spin model with long-range interactions,[8] as well as the diffusion of a passive scalar in a turbulent flow field.[9,10] Herein we stress the application to turbulent diffusion since its space-time context subsumes the characteristics of the less complex phenomena. We stress, however, that the arguments presented here are quite general and extend beyond the bounds of turbulent fluid flow.

Fully developed "homogeneous" turbulence involves spatial and temporal features covering many scales and no satisfactory description of the dynamics of such a flow field exists.[10] Three important developments which historically have furthered our incomplete understanding of turbulence are the following.

(i) Richardson's $\frac{4}{3}$ law: In an empirically motivated argument Richardson[9] constructed the phase-space equation of evolution

$$\partial P(R,t)/\partial t = (\partial/\partial R)[K(R)\partial P(R,t)/\partial R] \qquad (1)$$

with the turbulent diffusion coefficient given as $K(R) \propto R^{4/3}$, where $P(R,t)$ is the probability that two particles placed in a turbulent fluid and initially near to one another have a relative separation R at time t. An extended discussion of Eq. (1) and its degree of validity is given in Sect. 24 of Monin and Yaglom.[10] The solution to (1) leads immediately to Richardson's observation that $\langle R^2;t\rangle \sim t^3$, where the brackets denote an average over the distribution $P(R,t)$ which solves (1).[8,9]

(ii) Kolmogorov's $-\frac{5}{3}$ law: In 1941 Kolmogorov[11] argued that at small scales and in the limit of vanishing viscosity the rate of energy transfer across a scale R, ε_R, is independent of R, i.e., $\varepsilon_R = \bar{\varepsilon}$, a constant, in a region of active turbulence. This argument leads to an energywave-number spectrum $E(k) \sim \bar{\varepsilon}^{2/3} k^{-5/3}$. In 1962 Kolmogorov[12] acknowledged that turbulent dissipation is spatially dependent and proposed log-normal statistics for the velocity field to describe the observed patchiness in the turbulent activity.

(iii) Mandelbrot's intermittency: At least since 1974 Mandelbrot[13] has argued that turbulent activity (dissipation) is concentrated on a fractal set of dimension d_f, rather than being homogeneous in a Euclidean dimensionality $E=3$. This result was anticipated in part by Richardson,[9] who noted that the velocity field in the atmosphere shared a number of properties with the Weirstrass function, i.e., it appeared to be continuous but not differentiable. Recent experiments[14] suggest that $\mu = E - d_f \sim 0.2 \pm 0.05$.

Previous attempts to relate these turbulence results with the use of Lévy distributions have been unsatisfactory.[10] Although scaling is inherent to Lévy laws, no direct connection to turbulence is obvious because the mean square displacement $\langle R^2; t\rangle$ is infinite and not proportional to t^3 as found by Richardson.[9] In the next section we introduce a random-walk process which we call a *Lévy walk*. Based on Lévy-type distributions this walk results in enhanced diffusion for which $\langle R^2; t\rangle$ grows superlinearly with time (t^3 for turbulence). This Lévy-walk approach is a generalization of Taylor's correlated walks[15] which result in $\langle R^2; t\rangle \sim t^2$. We show the connections among Richardson's law, Kolmogorov's scaling, and Lévy walks as a context in which to develop a general theory of enhanced diffusion. The spatial intermittency corrections called for by Mandelbrot can be treated in a natural fashion to provide corrections to Richardson's law.

Consider a random walker who jumps with probability $p(\mathbf{R})$ between successively visited sites, however distant. When the mean square displacement $\langle R^2\rangle$ per jump is *finite* then the probability density for the position of the walker after many steps is Gaussian. When $\langle R^2\rangle$ is *infinite* this random process possesses no characteristic length scale and the set of sites visited is a fractal. This random process is called a Lévy flight and the governing probability density is called a Lévy stable law.[1,16]

Very often one considers that the walker can wait, or pause, for a random duration at each site before making an instantaneous jump to another site.[17] This waiting time can be interpreted to correspond to the jump distance.[18] Such processes have been called continuous-time random walks because the emphasis is on the time rather than the number of steps.[17] Here we extend these concepts and introduce $\Psi(\mathbf{R},t)$ to be the probability density of making a step $\mathbf{R}$ that takes a time t to complete, thereby generalizing the notion of a Lévy flight.

We now consider a random walker which visits the same sites as a random Lévy flight, but instead of having instantaneous jumps which lead to an infinite mean square displacement we choose the joint space-time

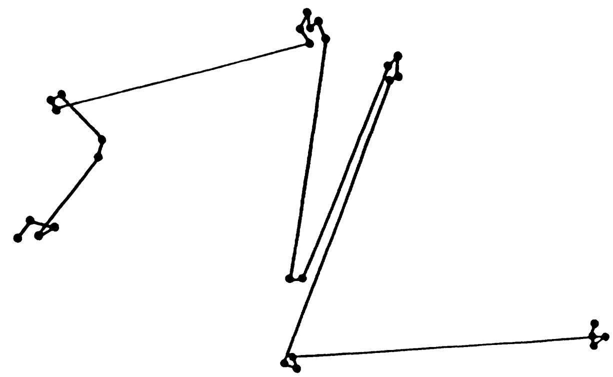

FIG. 1. The set of points visited by a Lévy walk. It includes the same set of points as the Lévy flight, plus the trail it takes connecting these points. The Lévy flight points are called turning points. Different velocities can be associated with different path lengths.

probability density

$$\Psi(\mathbf{R},t) = \psi(t \mid \mathbf{R}) p(\mathbf{R}), \tag{2}$$

where $p(\mathbf{R})$ is the probability that a jump (or correlated persistence length) of displacement $\mathbf{R}$ occurs and $\psi(t \mid \mathbf{R})$ is the conditional probability density that, given that the jump $\mathbf{R}$ occurs, it takes a time t to be completed. For simplicity we choose

$$\psi(t \mid \mathbf{R}) = \delta(t - |\mathbf{R}| / |\mathbf{V}(\mathbf{R})|), \tag{3}$$

where we take into account a velocity which depends on the jump distance (or length of the correlated persistence). When the second moment $\langle R^2\rangle$ of $p(\mathbf{R})$ in Eq. (2) is infinite then no largest walk scale exists and we call this process a Lévy walk to distinguish it from a Lévy flight. The distinction is introduced by Eq. (3) which weights the jumps according to the time spent in each step. We call the end points of each straight line segment in Fig. 1 the turning points of the trail of the Lévy walk. These are the points visited by the associated Lévy flight.

The probability density $Q(\mathbf{R};t)$ of reaching the turning point $\mathbf{R}$ exactly at time t is given by (in a lattice model)

$$Q(\mathbf{R},t) = \sum_{\mathbf{R}'} \int_0^t Q(\mathbf{R}-\mathbf{R}', t-\tau) \Psi(\mathbf{R}',\tau) d\tau + \delta(\mathbf{R}) \delta(t). \tag{4}$$

The probability $P(\mathbf{R},t)$ of being on any point $\mathbf{R}$ at time t is then obtained from

$$\frac{d}{dt} P(\mathbf{R},t) = \sum_{\mathbf{R}'} \int_0^t Q(\mathbf{R}-\mathbf{R}', t-\tau) W(\mathbf{R}',\tau) d\tau, \tag{5}$$

where

$$W(\mathbf{R}',\tau) \equiv \int_1^\infty \Psi(\lambda \mathbf{R}', \lambda \tau) d\lambda. \tag{6}$$

The Q term brings the walker to $\mathbf{R}-\mathbf{R}'$ at time $t-\tau$, while the W term allows for jumps of length $\lambda\mathbf{R}'$ to be completed in times $\lambda\tau$ for all $\lambda \geq 1$. All these jumps considered in the W term have average velocity $|\mathbf{R}'|/\tau$ and thus pass through $\mathbf{R}$ at time t; except for the $\lambda=1$ term the walk segments pass through but do not terminate at the point $\mathbf{R}$. Equation (5) generalizes the standard continuous-time random-walk approach wherein the walker pauses for random times between instantaneous jumps.[17] Equations (4)–(6) present a formulation of the problem in which the dynamics and the physical observables are described by an *integral* equation in which the multiplicity of scales is explicitly taken into account. This differs from the diffusion-equation approaches of Richardson[9] and Batchelor[19] which are inappropriate for describing a discontinuous or intermittent process such as turbulence.

Combining Eqs. (4) and (5) allows one to calculate $P(\mathbf{R},t)$ (and thus in principle all statistical quantities) in terms of $\Psi(\mathbf{R},t)$, i.e., in Fourier ($\mathbf{R}\to\mathbf{k}$) and Laplace ($t\to s$) space:

$$s\tilde{P}(\mathbf{k},s)-P(\mathbf{k},t=0)=1/[1-\tilde{\Psi}(\mathbf{k},s)]\tilde{W}(\mathbf{k},s). \qquad (7)$$

The mean square displacement of the process at time t is given by

$$\langle R^2;t\rangle=\int|\mathbf{R}|^2P(\mathbf{R},t)d\mathbf{R} = -L^{-1}\nabla_{\mathbf{k}}^2\tilde{P}(\mathbf{k},s)|_{\mathbf{k}=0}, \qquad (8)$$

where L^{-1} is the inverse Laplace transform. Note that Eq. (7) is quite general and can be applied to any physical problem for which the correlation of the steps [cf. Eq. (3)] is known.

We now consider a random-walk description of a fluid medium which possesses a wide distribution of correlation lengths and associated velocities generated by the spatial and temporal structure of fully developed turbulence. We use the Lévy walk as a statistical representation of this process and choose in Eq. (2) for large $|\mathbf{R}|$ the probability for a correlated persistence length of displacement $|\mathbf{R}|$ to be

$$p(\mathbf{R})\sim|\mathbf{R}|^{-1-\beta},\quad 0<\beta<1, \qquad (9)$$

so that the distribution of walk distances has no characteristic mean scale, i.e., walks of all lengths occur. We choose $\psi(t|\mathbf{R})$ as in Eq. (3) and below use Kolmogorov's scaling[11,12] to choose the proper dependence of the velocity on the length scale R.

Let us assume that the *relative* velocity $\mathbf{V}(\mathbf{R},t)=\mathbf{v}(\mathbf{r},t)-\mathbf{v}(\mathbf{r}+\mathbf{R},t)$, of two particles separated by a distance $\mathbf{R}$, is statistically independent of $\mathbf{r}$. For the case of turbulent diffusion the conditional probability that a jump of distance $\mathbf{R}$ occurs, taking a time t to be completed, is dependent on the relative velocity between two particles in the same active region of the fluid flow. Since this velocity is built up from a superposition of eddies, each with a distinct velocity, the rate of separation of the two marker particles depends on the distribution of relative velocities. If the turbulence is concentrated on a fractal set of dimension d_f, and the usual thermal molecular diffusion governs the much slower motion not on the fractal set, then the average kinetic energy density E_R associated with a scale R is $E_R\sim V_R^2p_R$, where p_R is the probability that the two particles in question are both on the fractal set. The probability p_R is given by $(R/R_0)^{E-d_f}$, where R_0 is an outer length scale. Benzi and Vulpiani[20] use a combination of the theory of stochastic differential equations and scaling to estimate p_R, whereas Hentschel and Procaccia[21] use a scaling argument alone. The rate of energy transfer across the scale R is $\varepsilon_R\approx E_R/t_R=V_R^3p_R/R$ when both $\mathbf{v}(\mathbf{r},t)$ and $\mathbf{v}(\mathbf{r}+\mathbf{R},t)$ are in the same *active region* of the flow. If $\varepsilon_R=\bar{\varepsilon}$ denotes a constant rate of energy transfer then

$$V_R\approx\bar{\varepsilon}^{1/3}R^{1/3}(R/R_0)^{-(E-d_f)/3};$$

and denoting the root-mean-square velocity as $V(R)$, we obtain from $V^2(R)\approx V_R^2p_R$ the result

$$V(R)\sim R^{\gamma},\quad \gamma=\tfrac{1}{3}+(E-d_f)/6. \qquad (10)$$

The scaling relation given by Eq. (10) enables us to calculate the probability density that a displacement of length R takes a given time t to be completed by use of Eq. (3). Inserting this value for $V(\mathbf{R})$ into Eq. (3) and using (9) to evaluate Eq. (8), we calculated the mean square separation of two particles in an active region of the flow as $t\to\infty$:

$$\langle R^2;t\rangle\sim\begin{cases} t^{12/(4-\mu)}=t^{3\mu/(4-\mu)+3}, & \beta\leq(1-\mu)/3,\\ t^{2+6(1-\beta)/(4-\mu)}, & (1-\mu)/3\leq\beta\leq(10-\mu)/6,\\ t, & \beta\geq(10-\mu)/6,\end{cases} \qquad (11)$$

where $\mu=E-d_f$. Note that the scaling exponents depend on the index β as well as the fractal dimension, a result not previously encountered and one which relates the spatial dynamics to the various diffusive regimes. It is interesting to point out that the time exponent in the first domain is also the value determined by Hentschel and Procaccia,[21] who used a much different method of analysis. Furthermore, the two limits in the second case smoothly match the first and third cases. The first case $[\beta\leq(1-\mu)/3]$ corresponds to an infinite mean time spent in a correlated transition so that no characteristic transition time exists. Diffusion is the most enhanced in this domain. The last case $[\beta\geq(10-\mu)/6]$ is analogous to molecular diffusion. Note further that if $\mu=0$, i.e., there is no correction for intermittency, and $\beta\leq\frac{1}{3}$ the Richardson t^3 law is recovered as we anticipated.

We have provided a statistical description of enhanced diffusion based on Lévy's probability limit distributions for random variables with infinite moments. A divergent result for the mean square displacement is avoided and replaced by a time-dependent result by our associating a time scale with correlation lengths (jump distances). The basic transport equation involves an *integral equation with a coupled (scaled) memory* $\Psi(\mathbf{R},t)$ nonlocal in space and time. This is in contrast to the approaches of Richardson[9] and Batchelor,[19] who used second-order differential equations with a space- and time-dependent diffusion coefficient to describe the enhanced diffusion of turbulence. Richardson himself pointed out the questionable nature of that approach because of the discontinuous nature of the velocity flow field in the atmosphere.[9] Foios, Manley, and Temam[22] used a simple random-walk model to interpret a new form of the Navier-Stokes equation they derived for which an intermittency exponent was also obtained. Using a Lévy-walk stochastic process, with a memory function based on Kolmogorov scaling, we derive Richardson's t^3 law of turbulent diffusion with the intermittency corrections called for by Mandelbrot.[13]

Within the framework of the Lévy-walk description of enhanced diffusion we conclude, according to Eq. (11), that Kolmogorov scaling does not necessarily imply Richardson's law. We do, however, derive the same results for $\langle R^2;t\rangle$ as previous authors[21] for the values $\beta \leq (1-\mu)/3$ which in the present context indicate that low-order moments do not uniquely determine the statistics of non-Gaussian processes. The Lévy-walk approach also provides a *dynamical* picture of the scalar motion, which will allow the study of other such dynamical processes including, for example, chemical reactions in turbulent flows with the use of known random-walk schemes for reaction rates via first-passage–time calculations.

If in our first analysis we had let $V(\mathbf{R}) = \bar{V}$ (a constant) then the result of Taylor[15] for maximal enhanced diffusion $\langle R^2;t\rangle \sim t^2$ would have been obtained. This result has also been obtained via a mapping[3] as well as via a Lévy walk to describe chaos in a Josephson junction.[4,23] It is thus apparent that fully developed *turbulence* (in a fluid) and *chaos* (in a Josephson junction) are quite different phenomena. However, a common element in the description of these processes is the Lévy-walk approach to enhanced diffusion.

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