Spinning Cosmic Strings and Quantization of Energy

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(Received 19 March 1986)

It is shown that energy should be quantized if there exist cosmic spinning strings. Quantization of energy arises naturally from the fact that the wave function must be single valued in the gravitational field of a spinning string. This effect arises when the vortexlike vacuum configurations carry constant nonzero angular momentum per unit length. Spinning strings occur naturally in spontaneously broken grand unified theories. A limit on J coming from the upper limit of the photon mass is presented. The analogy with quantization of energy in the presence of gravitational magnetic mass is established.

PACS numbers: 04.20.Jb, 12.10.Gq

The unusual properties of composite systems of charged particles and magnetic monopoles or vortices has attracted much interest since the celebrated paper by Dirac.[1-3] It was demonstrated in general that physical systems containing solitons and/or topological defects carry unexpected quantum numbers.[1-3] Dirac was the first who noticed that the quantization condition for electric charge emerges from the fact that wave functions must be single valued.[1] In recent years similar effects have been demonstrated in the framework of unified gauge theories.

On the other hand, the Aharonov-Bohm effect[4] shows that even if the gauge field is locally pure gauge, i.e., a curvature of the underlying gauge connection is (locally) vanishing, it can have a nontrivial effect on the quantum mechanical system. The phase of a wave function is nontrivially modified by such a gauge field. In other words, the existence of noncontractible loops can be detected quantum mechanically. The existence of pointlike or line "defects" like magnetic monopoles or vortices (strings) implies unusual angular momenquantization.

In general relativity, which in some sense is a gauge theory of gravity, it is the mass energy of a particle which plays the role of electric charge. It is natural, therefore, to ask if one can have quantization of energy in the spirit of Dirac or the gravitational analog of the Aharonov-Bohm effect.

Zee recently discussed a gravitational analog of Dirac's magnetic monopole which he calls "gravitipole."[5] His construction is based on the post-Newtonian approximation to the Einstein equations. It is well known that there exists an exact Taub-NUT (Newman-Unti-Tamburino) solution of Einstein's equations which corresponds to Zee's "gravitipole."[6] The Taub-NUT or asymptotically Taub-NUT metrics are characterized by the presence of gravitational magnetic mass N which is completely conserved (nonradiating) and as discussed first by Ashtekar and Sen[7] it has a topological origin. In classical general relativity those solutions are irrelevant because there is causality violation caused by the presence of magnetic mass. One can quantize fields in the background of an asymptotically Taub-NUT gravitational field[8] and obtain the quantization condition for energy.

The natural setting for discussion of the gravitational Aharonov-Bohm effect is three-dimensional gravity, where the vacuum Einstein equations $R_{\mu\nu}=0$ imply that curvature vanishes (locally), i.e., $R_{\mu\nu\alpha\beta}=0$. However, as it was demonstrated over twenty years ago by Staruszkiewicz[9] and recently rediscovered by Deser, Jackiw, and 't Hooft (and others),[9] this theory is globally nontrivial. Classically, in this theory the equations of motion of particle singularities (as well as of extended objects first studied by Staruszkiewicz) are implied exactly by the field equations as opposed to the four-dimensional (4D) case. The absence of the graviton in three dimensions makes it possible that the model is exactly solvable.

It is obvious that one can obtain interesting solutions to 4D gravity by reversing the dimensional reduction procedure starting with solutions of 3D gravity and lifting them to a trivial $R\times M_3$ R^1 bundle over a 3D manifold M_3. In other words, assuming that the 4D solution possess a Killing vector with a constant norm and zero "twist," one has ${}^{(4)}R={}^{(3)}R$, i.e., the Einstein-Hilbert Lagrangean density in 4D reduces to the three-dimensional one. In this way one obtains locally flat 4D solutions with line singularities. The three-dimensional "Kerr" solution describing the gravitational field of a massive and spinning particle[9] corresponds in 4D to the solution describing a spinning string with a constant mass and angular momentum per unit length. The gravitational field produced by an infinitely "thin" nonspinning string has been studied recently.[10-12]. Static and cylindrically symmetric string solutions to the Abelian Higgs model coupled to gravity[13] as well as a spinning (stationary) solution to the non-Abelian Higgs model coupled to gravity have been obtained recently.[14] In the limit of thin flux tubes one recovers the locally trivial gravitational field of the string, i.e., the space-time is locally flat and has a conical geometry. Paranjape[15] has shown that in the presence of a magnetic flux tube, the ground state of Dirac fermions carries a net angular momentum. The exciting possibility of superconducting cosmic strings where the charge carriers are left- and right-moving fermionic zero modes has been proposed by Witten.[16] These strings can have also a net angular momentum carried by fermionic modes. So the existence of spinning strings is possible.[14]

It is the purpose of this Letter to discuss the quantum mechanical properties of massive (or massless) particles in the gravitational field of spinning strings. I will show that the physical effect of nonzero angular momentum per unit length of string is analogous to the effect of magnetic charge. In the presence of a spinning string the energy of a particle is quantized according to the Dirac-type formula $E=\hbar c^4/4GJ$, where G is the Newton constant and J is angular momentum per unit length.

The spinning string solution and quantization of energy.—The three-dimensional "Kerr solution"[9] corresponds to a time-independent (stationary) spatially localized spinning source with nonzero energy and angular momentum density. The angular momentum in $D=2$ (spatial) is a pseudoscalar,

$$J=\tfrac{1}{2}\epsilon_{ij}J^{ij}=\tfrac{1}{2}\epsilon_{ij}\int d^2x\,(x^iT^{0j}-x^jT^{0i}). \qquad (1)$$

The metric found by Deser, Jackiw, and 't Hooft[9] has a simple form:

$$ds^2_{(3)}=-(dt-A\,d\phi)^2+d\rho^2+\rho^2\alpha^2\,d\phi^2, \qquad (2)$$

where $\alpha=1-4Gm$ and m is a mass of a particle. The value of J^{ik} is $J^{ik}=-(4G)^{-1}A\epsilon^{ik}=J\epsilon^{ik}$. The constant A is determined to be $-4GJ$. The corresponding spinning-string solution is therefore

$$ds^2_{(4)}=-(dt+4GJ\,d\phi)^2+\rho^2a^2\,d\phi^2+d\rho^2+dz^2, \qquad (3)$$

where J and m are the string angular momentum and mass per unit length. The metric (3) is locally flat and can be transformed into Minkowski form,

$$ds^2=-dT^2+dX^2+dY^2+dZ^2, \qquad (4)$$

by a coordinate transformation

$$T=T+2GJa^{-1}\phi',\quad \phi'=a\phi,$$
$$X=\rho\cos\phi',\quad Y=\rho\sin\phi',\quad Z=z. \qquad (5)$$

This coordinate transformation is singular at $\rho=0$. One can adopt two possible interpretations of the string metric (3): (i) The metric is locally flat with periodic time coordinate with the period $8\pi GJ$. (ii) The time is not a periodic coordinate, but there is a causality violation in the region $\rho<\rho_0$. The quantization-of-energy condition is true irrespective of the interpretation adopted for the time variable. We have a singular line source producing this kind of gravitational field.

Even if the metric (3) is locally flat there is a singularity present. Namely, at constant t as ϕ reaches 2π (which should be identified with $\phi=0$), T jumps by $8\pi GJ$. This "time-helical" structure of the spinning-string space-time has interesting quantum-mechanical consequences. Deser, Jackiw, and 't Hooft suggested that one has to identify times t which differ by $8\pi GJ$ to preserve single valuedness and smoothness of the manifold. When one quantizes fields in the gravitational field of a string one will have fractional angular momentum, $j=a^{-1}\hbar n$, n a half integer. The angular momentum spectrum is altered by a factor a^{-1} because the space-time has locally flat conical geometry, i.e., the range of the angular variable ϕ' is $2\pi\alpha$.

A way of measuring the mass per unit length m would be through the gravitational Aharonov-Bohm effect.[4] The locally vanishing curvature would be reflected in the phase of a quantum particle. In the geometrical-optics approximation one can see the phase shift by studying parallel transport of vectors (i.e., four-momentum, polarization vector, etc.) along closed paths. One may study the change of polarization vector of a high-energy photon moving along two different paths containing a string inside. This effect is small and is of order Gm, as is everything in the string background, because the only available parameter is $\alpha=1-4Gm$. The effect of angular momentum J can be probably seen in a sort of Sagnac effect.[17] Now I will show that the interpretation of T as an anglelike variable with period $8\pi GJ$ is forced by quantum mechanical principles.

Consider the Klein-Gordon equation for a particle with mass μ in the gravitational field of the spinning-string solution:

$$(-\nabla_\alpha\nabla^\alpha+\mu^2)\phi=0. \qquad (6)$$

We are looking for solutions of the form

$$\phi=e^{-iET/\hbar}e^{ikZ}\psi(\rho,\phi)$$
$$=e^{-iEt/\hbar}e^{ikZ}e^{-i(4GJE/\hbar)\phi}\psi(\rho,\phi), \qquad (7)$$

where $\psi(\rho,\phi)$ is a single-valued function of ϕ. As ϕ increases from 0 to 2π and T "jumps" by $8\pi GJ$, the wave function ϕ acquires the phase $e^{-8\pi iGJE/\hbar}$. We conclude, following Dirac's argument, that $E=n\hbar c^4/4GJ$, n an integer. Energy is quantized in units of $\hbar c^4/4GJ$. It does mean that t (or T) is an anglelike variable with period $8\pi GJ$.

Before discussing another example of energy quantization, let me present another derivation of the spinning-string solution. We are looking for stationary, cylindrically symmetric vacuum (exterior) solutions to Einstein's equations of the form[17]

$$ds^2 = -e^{2\nu}dt^2 + e^{2\psi}(dt - \omega\, d\phi)^2 + e^{2\mu}(dr^2 + dz^2), \tag{8}$$

where ν, ψ, and μ are functions of r only. The derivation of solutions is straightforward and I refer to Ref. 7 for the form of field equations for (8). In the case of rigid rotation, i.e., $\omega = \text{const}$, which leads us to interesting solutions (the nonrigid rotation case has pathological solutions and we will not discuss them here), the most general exterior solution is the rotating Levi-Civita metric,

$$ds^2 = -r^{2a}(dt - A\, d\phi)^2 + a^2 r^{2(1-a)}d\rho^2 + r^{2a(a-1)}(dr^2 + dz^2). \tag{9}$$

When $A = 0$ this is the Levi-Civita metric.[18] But nonspinning strings are invariant with respect to Lorentz boosts along the z-axis. This restricts the metric of the nonrotating string, and we have two solutions with $a = 0, 2$ (also for a spinning string). The case $a = 2$ is ruled out as "pathological" because as $r \to \infty$ the space-time becomes effectively three-dimensional. Eventually we arrive at the solution found previously by extending to four dimensions the solution of 3D gravity. At this point it should be noted that one can obtain stationary, cylindrically symmetric solutions of the Einstein-Yang-Mills-Higgs model with the gauge group SO(3), which describe spinning grand-unified-theory strings. The solutions do exist and can be evaluated numerically for arbitrary Higgs coupling constant λ. In the Prasad-Sommerfield limit $\lambda = 0$, the solution is known analytically.[14]

Quantization of energy: analogy with electromagnetism.—I shall present a simple general-relativistic example which makes the analogy between quantization of energy and Dirac's quantization of electric charge more transparent. Consider the stationary metric, which can be seen as the asymptotic form of some solutions of Einstein's equations:

$$ds^2 = -V(dt - A_i\, dx^i)^2 + h_{ij}\, dx^i dx^j, \tag{10}$$

where V, A_i, and h_{ij} are functions of x^i only. The allowed "gauge" transformations are $t \to t' = t + \Lambda(x^i)$ and the diffeomorphisms of $t = \text{const}$ hypersurfaces. The "gauge" transformation implies that A_i transforms like the electromagnetic gauge connection, $A_i \to A_i' = A_i + \nabla_i\Lambda$. The three-metric h_{ij} is asymptotically (locally) Euclidean. The notion of quantum particle in a gravitational field is well defined only asymptotically, i.e., in the asymptotically flat region. We have then the asymptotic positive-frequency modes defined with respect to time t and one can construct the Fock space of asymptotic states. The metric (10) is asymptotically flat by construction.

I consider two examples by taking for A_i the Dirac magnetic-monopole gauge field and the gauge field produced by a thin, infinitely long solenoid (vortex) (the Aharonov-Bohm gauge field). The first example corresponds to the asymptotic Taub-NUT space-times[7] which are characterized by nonvanishing magnetic mass. The second one corresponds to the spinning-string metric. Let A_i be the Dirac magnetic-monopole field $A_\phi = -2N(1-\cos\theta) = -4N\sin^2\theta/2$. In this case the metric is spherically symmetric (if h_{ij} is flat or spherically symmetric) and has a coordinate singularity at $\theta = \pi$ ("string" singularity of the gauge field). Now, if we consider the Klein-Gordon equation, the wave function (section) has a form $\phi = e^{-iEt/\hbar} \times\psi(r,\theta,\phi)$. As usual we have to introduce two coordinate patches on the space-time manifold because A_i cannot be defined globally on S^2 if there is a magnetic monopole. One should also consider a wave section[19] for a scalar field. In the gauge where A_i is regular at $\theta = \pi$ we have

$$A_\phi' = +2N(1+\cos\theta) = A\phi + 4N$$
$$= A\phi + \partial\phi\Lambda, \quad \Lambda = 4N\phi. \tag{11}$$

This corresponds to a coordinate transformation $t' = t + 4N\phi$ and the wave function (section) ϕ changes to $\phi' = e^{-iEt'/\hbar}\psi'(r,\theta,\rho)$, where ψ' is a single-valued function of ϕ. If we demand that ϕ' is single valued then the Dirac-type quantization condition emerges naturally: $4NE = \hbar n$, n an integer. The quantum mechanical consistency conditions once again force us to interpret the time t as an angular variable with period $8\pi N$. The space-time acquires the topology $S^3\times R$ with the vector field $\partial/\partial t$ generating the S^1 (Hopf) fibers of S^3. The coordinates on S^3 are $\psi = t/2N$, θ, and ϕ. These are closed timelike curves, and we have unfortunately causality violation implied by the presence of magnetic mass. In the case of a k-monopole connection A_i the null infinity has topology of the lens space $L(k,l)$ which is S^3 with Z_k identifications made along S^1 (Hopf) fibers.[7] It is the asymptotic structure of a metric which is relevant in the derivation of the quantization condition of energy but not its local behavior at "short distances." One can say, therefore, that our result is purely "kinematic," because it does not depend on the field equations to which these metrics are solutions. At this point one should note that the post-Newtonian construction of "gravitipoles" by Zee[5] corresponds to our simple example. However, our argument for quantization of energy is completely general-relativistic in contrast to the nonrelativistic mechanical argument by Zee. Therefore not only is the energy of massive particles quantized, but also this is true for massless particles like the photon. The Planck relation $E=\hbar\omega$ relating frequency to energy is also valid for quantum systems in the field of a "gravitipole," which seemed to be an unclear point in Ref. 5.

The second example corresponds to the spinning-string solution I have discussed earlier. The potential A_i for this case is that of the Aharonov-Bohm effect, $A_\phi = A = \text{const}$, and can be locally gauged away by the singular gauge transformation $\Lambda = -A\phi$. It does mean that the time is transformed to $t' = t - A\phi$, and the wave function $\phi = e^{-iEt/\hbar}\psi(\rho,z,\phi)$ gauges to

$$\phi' = e^{-iEt'/\hbar}\psi'(\rho,z,\phi) = e^{iAE\phi/\hbar}e^{-iEt/\hbar}\psi'(\rho,z,\phi).$$

Once again single valuedness of ϕ' implies the quantization condition $EA = \hbar n$. The proper approach to formulation of quantum mechanics in the cases discussed here is the formulation in terms of wave sections and complex vector bundles.[19]

The metric of the spinning string has causality-violating regions as can be seen from the fact that the norm of the rotational Killing vector $\partial/\partial\phi$ (which has closed orbits) changes sign at $\rho = \rho_0 = 4GJ\alpha^{-1}c^{-3}$. For $\rho < \rho_0$ there are closed timelike curves which are orbits of $\partial/\partial\phi$. One can estimate the size of the causality-violating region knowing how large J can be. From quantization of energy E one can estimate J knowing what is the lowest quantum of energy $E_{\min}$. The natural constraint on $E_{\min}$ emerges from the limit on photon mass, $E_{\min} \sim 10^{12}$ eV. This gives $J = \hbar c^4/4GE \sim 3\times 10^{33}$ erg sec/cm. Also, a periodicity of time of 6×10^{-3} sec is not encouraging. For this resaon one should not expect spinning strings to be realistic or stable. One could imagine that once they were formed they might decay because of instabilities. Concluding, I have shown that quantization of energy emerges naturally when there exist spinning cosmic strings. My argument shows that energy is quantized for purely topological reasons.

If the string is not infinitely thin and we have the region of nonrigid rotation $\omega = \omega(r)$ for $r < r_0$ matching a region of rigid rotation $\omega = \omega_0$, $r > r_0$, where r_0 is the string thickness, then the asymptotic form of the metric (8) is given by Eq. (3). The previous arguments about quantization of energy are unchanged. The situation here is quite similar to the case of the Dirac quantization condition which is the same for the Dirac or 't Hooft–Polyakov monopole solution.

I would like to thank Abhay Ashtekar for many discussions and for encouraging me to write up this work, and Stanley Deser for useful conversations. This work was supported in part by the National Science Foundation through Grant No. PHY 8318350 with Syracuse University.

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