Trajectories of Strings with Rigidity

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Classical solutions of elastic strings (i.e., strings with actions which depend on the extrinsic curvature of the world sheet) are studied to determine Regge trajectories. For open strings, our classical solutions are identical to those for the conventional Nambu-Goto action. However, for closed strings, new solutions give nonlinear trajectories that include finite-energy, static configurations.

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Polyakov[1] has recently focused attention on actions for relativistic strings which depend on the extrinsic curvature of the string's world sheet as embedded in higher-dimensional space-time. Such "elastic" string actions contain quartic derivatives and may appear in effective actions of strings as a result of one's functionally integrating out fermions.[2] Polyakov noted that the inverted coefficient for the extrinsic curvature action—the rigidity—is asymptotically free. These perturbative renormalization effects had previously been investigated in other contexts.[3] For example, it was known that lipid membranes with small surface tension have their effective rigidity reduced for long wavelengths because of thermal fluctuations in the fluid in which the membranes are immersed.

Interactions which are sensitive to the extrinsic properties of the world sheet's space-time embedding will influence the compactification of extra spatial dimensions. In addition to controlling a "crumpled phase" of strings,[1] these interactions should suppress longitudinal kink/fold modes,[4] and they may be a desirable feature for Pomeron phenomenology.

In this Letter we explore the classical solutions of the string equations of motion when rigidity terms are present. We determine Regge trajectories which represent the J (angular momentum) versus E^2 relationship for the classical string solutions. For the open string, the lowest-energy motion is the usual straight-line, pinwheel motion[5] familiar from the conventional, pliable, Nambu string. Extrinsic curvature vanishes for these motions. However, for the closed string there is another motion which supplants, for low J values, the standard folded-over pinwheel rotation of the Nambu string. This new motion is the rotation of an oblate closed loop, which reduces to a circle in the static limit, with finite energy at $J=0$. For $J\neq 0$, the rotation rate for this string configuration first increases and then slows down again as E and J increase monotonically. For very high J values, the configuration elongates, and as ω again goes to zero, a limit of two infinite, parallel straight lines (with infinite energy) is approached.

Recall the Nambu-Goto "area-law" action in second-order form:

$$I_1 = -T_0 \int d^2\xi \sqrt{-g}, \quad g_{ab} = \partial_a X^\mu \partial_b X_\mu, \tag{1}$$

where T_0 is the tension, ξ^a $(a,b=0,1)$ are the world-sheet parameters, and X^μ $(\mu = 0, \ldots, D-1)$ are the space-time string coordinates. The corresponding equation of motion is the covariant wave equation

$$\begin{aligned} &T_0\sqrt{-g}\,\Box X^\mu = 0, \\ &\Box X^\mu = g^{ab} D_a D_b X^\mu \\ &\quad = (1/\sqrt{-g})\partial_a(\sqrt{-g}\, g^{ab}\partial_b X^\mu). \end{aligned} \tag{2}$$

Since I_1 depends only on the metric g_{ab} of the world sheet, it is sensitive only to the intrinsic geometry of the sheet and is impervious to the extrinsic curvature of the sheet. Thus, for example, I_1 does not distinguish between flat and corrugated sheets.

However, one may contemplate[1-3] strings with interaction terms depending on the derivatives of the sheet tangents (*vielbeine*), $\partial_a X^\mu$, through the second fundamental form[6]: $K^i_{ab} = n^i_\mu \partial_a \partial_b X^\mu$. Here n^i_μ are the $D-2$ unit normals to the sheet: $n^i_\mu n^{j\mu} = \delta^{ij}$, $n^i_\mu \partial_a X^\mu = 0$, $i = 1, \ldots, D-2$. In general, the Gauss-Weingarten formulas give the complete gradients of the *vielbeine*,

$$\partial_a \partial_b X^\mu = \Gamma^c_{ab} \partial_c X^\mu + K^i_{ab} n_i{}^\mu, \tag{3}$$

including the components tangent to the sheet, as in $\Gamma^c_{ab} g_{cd} = \partial_a \partial_b X^\mu \partial_d X_\mu$. The gradients of the normals also split into corresponding components: $\partial_a n^i_\mu = -(n^i_\nu \partial_a n^{j\nu}) n^j_\mu - K^i_{ab} g^{bc} \partial_c X_\mu$. Using these relations, we rewrite the covariant wave equation (2) as

$$\Box X^\mu = g^{ab} K^i_{ab} n^{i\mu}. \tag{4}$$

The new extrinsic-curvature–dependent addition to the action is

$$\begin{aligned} I_2 &= S_0 \int d^2\xi \sqrt{-g}\, (\Box X^\mu)^2 \\ &= S_0 \int d^2\xi \sqrt{-g}\, (g^{ab} K^i_{ab})^2. \end{aligned} \tag{5}$$

The coupling S_0 is called the "rigidity," since it opposes curving of the world sheet in the envelope space-time.

The action for the elastic string is the sum I_1+I_2, or

$$I=-S_0\int d^2\xi\sqrt{-g}\,[1/R_0^2-(\Box X^\mu)^2], \tag{6}$$

$R_0^2=S_0/T_0$, where we have defined a radius R_0. (S_0 is dimensionless, while T_0 is not.) By construction, I is world-sheet–reparametrization invariant.

We now consider classical motions of the string. First we establish a lemma. Since I_2 is quadratic in $\Box X^\mu$, solutions of the covariant wave equation for the Nambu string, (2), receive no modifications from I_2, including boundary conditions, provided that the metric is nonsingular, since additional derivatives of g_{ab} and $\Box X^\mu$ appear in the wave equation resulting from I.

Lemma.—All nonsingular classical solutions for the Nambu action are also solutions for I, with the same values for conserved quantities.

For convenience, we henceforth work in the timelike orthogonal gauge exploiting world-sheet reparametrization invariance to choose

$$X^0=\xi^0,\quad \dot{\mathbf{X}}\cdot\mathbf{X}'=0, \tag{7}$$

where $X^\mu=(X^0,\mathbf{X})$, $\dot{\mathbf{X}}=\partial\mathbf{X}/\partial\xi^0$, $\mathbf{X}'=\partial\mathbf{X}/\partial\xi^1$. In this gauge, the sheet metric reduces to

$$g_{ab}=\begin{pmatrix}1-\dot{\mathbf{X}}\cdot\dot{\mathbf{X}} & 0\\ 0 & -\mathbf{X}'\cdot\mathbf{X}'\end{pmatrix}, \tag{8}$$

while the action becomes

$$I=-S_0\int d^2\xi|\mathbf{X}'|(1-\dot{\mathbf{X}}^2)^{1/2}\left[\frac{1}{R_0^2}-\left(\frac{\dot{\mathbf{X}}\cdot\ddot{\mathbf{X}}}{(1-\dot{\mathbf{X}}^2)^2}+\frac{\mathbf{X}'\cdot\dot{\mathbf{X}}'}{\mathbf{X}'^2(1-\dot{\mathbf{X}}^2)}\right)^2\right.$$
$$\left.+\left(\frac{\ddot{\mathbf{X}}}{(1-\dot{\mathbf{X}}^2)}+\frac{\dot{\mathbf{X}}\mathbf{X}'\cdot\dot{\mathbf{X}}'}{\mathbf{X}'^2(1-\dot{\mathbf{X}}^2)}+\frac{\dot{\mathbf{X}}\dot{\mathbf{X}}\cdot\ddot{\mathbf{X}}}{(1-\dot{\mathbf{X}}^2)^2}-\frac{\mathbf{X}''}{\mathbf{X}'^2}+\frac{\mathbf{X}'\mathbf{X}'\cdot\mathbf{X}''}{\mathbf{X}'^4}+\frac{\mathbf{X}'\dot{\mathbf{X}}\cdot\dot{\mathbf{X}}'}{\mathbf{X}'^2(1-\dot{\mathbf{X}}^2)}\right)^2\right]. \tag{9}$$

The simplest string motions to consider are uniform "rigid-body" rotations about a fixed axis, $\hat{\boldsymbol{\omega}}$, with angular frequency ω. The *Ansatz* for such motions ($\boldsymbol{\omega}=\omega\hat{\boldsymbol{\omega}}$) is $\dot{\mathbf{X}}=\boldsymbol{\omega}\times\mathbf{X}$, $\mathbf{X}\cdot\dot{\mathbf{X}}=\dot{\mathbf{X}}\cdot\ddot{\mathbf{X}}=\mathbf{X}'\cdot\dot{\mathbf{X}}'=0$, $\dot{\mathbf{X}}^2=\omega^2\mathbf{X}^2-\boldsymbol{\omega}\cdot\mathbf{X}^2$. The action then simplifies to an expression linear in the time: $I=tL$, where $t=\int d\xi^0$. The angular momentum also simplifies considerably for such uniform motion since $\delta\mathbf{X}=\dot{\mathbf{X}}/\omega$. In fact, when the configuration of the string lies in a plane together with the axis of rotation, J is simply $dL/d\omega$. For such a rotating configuration, the energy also reduces to the standard Legendre transform of the action. Thus

$$J=dL/d\omega,\quad E=\omega J-L. \tag{10}$$

It is straightforward to verify that the open-string solution of the Nambu action, the rotating "pinwheel" motion,[5] is also a solution of the equations of motion resulting from (9), and that it satisfies the same boundary conditions at the ends of the string. The lemma holds without qualifications:

$$\mathbf{X}=\xi^1\hat{\mathbf{e}},\quad -1/\omega\le\xi^1\le 1/\omega \text{ (open string)}, \tag{11}$$

with $\hat{\mathbf{e}}\perp\hat{\boldsymbol{\omega}}$ and $\dot{\hat{\mathbf{e}}}=\boldsymbol{\omega}\times\hat{\mathbf{e}}$. The ends of the string therefore move at the speed of light, as for the Nambu string. The Lagrangean, energy, and angular momentum are given by

$$L=-\pi T_0/2\omega,\quad E=\pi T_0/\omega,$$
$$J=\pi T_0/2\omega^2=E^2/2\pi T_0. \tag{12}$$

Thus classical Regge trajectories for these open-string configurations are linear with slope $(2\pi T_0)^{-1}$.

For the closed-string case, however, the folded-over straight-line configuration (which extremizes the Nambu action and gives a Regge trajectory with half the slope of the open-string case) is *not* a solution. The above lemma is obviated in this case by the creased ends of the straight-line segments where the metric is singular. If the ends are rounded out slightly, with a radius ϵ, the rigidity term I_2 contributes $O(1/\epsilon)$ to the total action[7] [near the folded ends the Lagrangean goes like $\epsilon^2\int d\sigma\,\sigma^4/(\sigma^2+\epsilon^2)^4$]. Consequently this configuration for the closed elastic string is not a classical solution as $\epsilon\to 0$.

The actual, planar, classical solution for the rotating closed elastic string is easy to visualize since it has a nonrelativistic limit, unlike the Nambu string. It *resembles* an ellipse rotating about its minor axis. In this oblate hooplike configuration, the rigidity and centripetal acceleration balance the string tension. We may parametrize the solution by working in the plane of the hoop, defining r to be the distance from the center of the hoop, and θ to be the angle from the major axis: $\mathbf{X}=(r(\theta)\cos\theta,\ r(\theta)\sin\theta)$. With this parametrization, the Lagrangean for the hoop *Ansatz* reduces to

$$L=-S_0\int d\theta[r^2+(dr/d\theta)^2]^{1/2}[1-\omega^2r^2\cos^2\theta]^{1/2}\left[\frac{1}{R_0^2}+\left(\kappa-\frac{\omega^2 r\cos\theta[r\cos\theta+(dr/d\theta)\sin\theta]}{[r^2+(dr/d\theta)^2]^{1/2}[1-\omega^2r^2\cos^2\theta]}\right)^2\right], \tag{13}$$

where κ is the curvature for a planar curve,

$$\kappa = \frac{1-(d/d\theta)\arctan(r^{-1}dr/d\theta)}{[r^2+(dr/d\theta)^2]^{1/2}}. \tag{14}$$

Now it is easy to determine the static ($\omega=0$) solution for the hoop:

$$L(\omega=0) = -S_0\int d\theta\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]^{1/2}\left[\frac{2\kappa}{R_0}+\left(\kappa-\frac{1}{R_0}\right)^2\right]. \tag{15}$$

The first term on the right-hand side (RHS), linear[8] in κ, gives a topological invariant (Hopf's circulation theorem for closed planar curves[6]), which is clear from the explicit form for κ in (14). This term counts the number of times N the tangent to the curve rotates through 2π as the closed string is circumambulated. The second term on the RHS of (15) is obviously extremized (0) for a constant $\kappa=1/R_0$. This means that the static hoop forms a circle of radius $r=R_0$. For such a circle, wound N times with string, (15) immediately gives the action, and energy since $E=-L(\omega=0)$ in the static case:

$$E_N = 4\pi NS_0/R_0 = 4\pi N(S_0T_0)^{1/2}. \tag{16}$$

For the nonstatic ($\omega\neq 0$) case, we do not have similar topological/geometrical arguments to obtain the classical solution. First, we performed a small-ω perturbative analysis by substituting a Fourier series expansion for $r(\theta)$ and then varying the coefficients to extremize L. The result is

$$r(\theta) = R_0\{1+\omega^2R_0^2[1+\tfrac{1}{3}\cos(2\theta)]+O(\omega^4)\}. \tag{17}$$

Thus a slow rotation stretches both the major and minor axes. Substituting this result into L, and evaluating the angular momentum and energy, we obtain

$$\omega J = 6\pi NS_0R_0\omega^2+O(\omega^4),$$
$$E = 4\pi N(S_0/R_0)[1+\tfrac{3}{4}\omega^2R_0^2+O(\omega^4)], \tag{18}$$

which is not surprising for a nonrelativistic rigid-body rotation. The resulting Regge trajectories are nonlinear for small ω:

$$J = [\tfrac{3}{2}R_0^2(E^2-16\pi^2N^2S_0T_0)]^{1/2}+O(\omega^4). \tag{19}$$

We have numerically investigated the large-ω corrections to the trajectory segments specified by (19). For this purpose it is convenient to change variables from θ to a modified angle ϕ. Define

$$u=\cos\phi,\quad \phi=\theta-\arctan(r^{-1}dr/d\theta)=\int_0^\theta d\theta[r^2+(dr/d\theta)^2]^{1/2}\kappa. \tag{20}$$

Clearly, $u=1$ ($\phi=0$) corresponds to a point on the hoop which lies on the major axis, while $u=0$ ($\phi=\pi/2$) corresponds to a point on the minor (rotation) axis. We also switch to a dependent variable g, which is the distance from the axis of rotation. Using (14), we have $g=r(\theta)\sin\theta$, $dg/du=1/\kappa$. With these variables, the *Ansatz* rotating hoop system is described by the Lagrangean

$$L = -4S_0\int_0^1 du\left(\frac{1-\omega^2g^2}{1-u^2}\right)^{1/2}\frac{1}{\kappa}\left[\frac{1}{R_0^2}+\left(\kappa-\frac{\omega^2ug}{1-\omega^2g^2}\right)^2\right], \tag{21}$$

where we have assumed a reflection symmetry for the solution ($u\to -u, g\to -g$). The angular momentum and energy for the *Ansatz* are again given by (10). Varying g in L leads to the Euler equation

$$\frac{d}{du}\left(\kappa-\frac{\omega^2ug}{1-\omega^2g^2}\right) = \frac{u}{1-u^2}\frac{1}{\kappa}\left[\frac{1}{R_0^2}-\kappa^2+\left(\frac{\omega^2ug}{1-\omega^2g^2}\right)^2\right], \tag{22}$$

which is a second-order, nonlinear equation for $g(u)$, since $\kappa=(dg/du)^{-1}$.

We solved Eq. (22) numerically as a boundary-value problem. Starting with $g(u=0)=0$, we integrated to the most rapidly moving point on the hoop, at $u=1$, where we required the boundary condition that $d\kappa/du$ remain finite. This required the numerator on the right-hand side of (22) to vanish. Alternatively, we calculated the action (21) as a function of an initial curvature $\kappa(u=0)$, with $g(u=0)=0$, and found the extrema. The two results agree within numerical uncertainties, although the latter action method appears to be better.

The results for the trajectory are shown in Fig. 1. For the same ω, there are in fact two extrema of L, and hence two classical solutions. One branch of solutions originates with the static circular hoop and develops for small ω as described above in (17), (18), and (19). Both $E(\omega)$ and $J(\omega)$ increase monotonically for this branch up to a critical value ω_c ($\simeq 3.5\sqrt{T_0}$, for $S_0=1$). At this critical value the action, con-

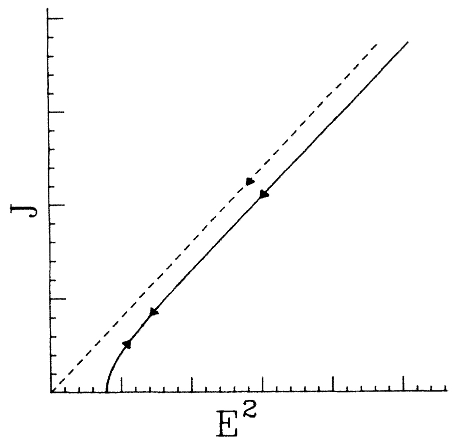

FIG. 1. The leading ($N=1$) trajectory for $S_0=1$ (solid line) and for $S_0=0$ (dashed line). Arrows indicate increasing ω.

sidered as a function of $\kappa(u=0)$, has an inflection point. The trajectory continues to higher E and J values through the second branch of solutions to (22), allowing ω to decrease back to zero.

The branch of solutions giving the upper segment of the trajectory behaves somewhat similarly to the motions of the Nambu string. As $\omega \rightarrow 0$, the major axis of the hoop grows as $1/\omega$, and the trajectory becomes linear. Unlike the Nambu closed string, however, the ends of this configuration are not sharp folds. The elastic energy stored in these ends displaces the trajectory to the right of the Nambu straight-line trajectory, as shown in Fig. 1.

As energy and angular momentum are pumped into the system, both the major and minor axes of the configuration grow, although the ratio decreases to zero. Correspondingly, the edges of the hoop farthest from the axis of rotation move with increasing speed. This increase continues smoothly from the lower-branch trajectory into the upper one, except that on the upper branch the growth of the edge speed lags the growth of the major-axis length. The critical frequency ω_c represents the maximum frequency attained by the solution. Since $\omega = dE/dJ$, the critical point ω_c also represents an inflection point with $d^2E/dJ^2=0$.

As S_0 is decreased, tending to the limit of the pliable string ($S_0=0$), the E^2 intercept of the trajectory moves toward the origin. So does the critical point, so that only the upper solution branch survives. This reduces to the straight-line trajectory which characterizes the Nambu string.

To summarize, for the closed-string sector, the trajectories are nonlinear and are dominated at low energies (short strings) by the novel hoop configurations discussed above. At higher energies (long strings), the relevant configurations exhibit more conventional Regge behavior, although classically they lack the sharply folded ends of the Nambu string because of the rigidity which discourages bending of the string.

This system must be quantized. Upon quantization, we expect a shift in the vacuum energy and some modifications of the low-energy spectrum. However, since the lowest-energy state of the classical closed string is an adjustable parameter [as in (16)], we expect at least for some range of S_0 that there will be no massless states for the quantized rigid string, and hence there will be no graviton in the model. This raises an issue of general covariance in the embedding space-time. Finally, the analytic continuation of the nonlinear trajectories to negative J and E^2, together with the resolutions of the intercepts of the sister trajectories [described classically in (16)], may have interesting implications in hadronic applications of string theory. It is also interesting to explore the relationship of the nonlinear trajectories presented here have with those described in the earlier literature.[9] Work on these questions is in progress.

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