Derivation of the Equilibrium Degree of Polarization in High-Energy Electron Storage Rings

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A semiclassical approach is used to derive, and extend to first order in $g-2$, the equilibrium degree of polarization in high-energy electron storage rings (the Derbenev-Kondratenko formula). Statistical concepts are shown to be essential for an understanding of this phenomenon. In so doing, some aspects of the polarization mechanism not previously recognized are uncovered.

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It was predicted many years ago that electrons and positrons in high-energy storage rings would become polarized by the emission of spin-flip synchrotron radiation; this is now known as the Sokolov-Ternov effect.[1] Some time ago, Derbenev and Kondratenko[2] gave a detailed formula for the equilibrium degree of polarization. Many workers have had difficulty in understanding these classic papers, partly due to their sophisticated mathematics but also because of the compact presentation. This Letter presents a rederivation[3] of the Derbenev-Kondratenko formula obtained by following Schwinger's[4,5] semiclassical techniques for calculating the synchrotron-radiation power spectrum. The formula is also extended to first order in $g-2$, and the previously uncertain consequences thereof are elucidated. This analysis does not simplify the mathematics but yields new insights into electron-spin polarization at high energies. In particular, an important point, which seems not to have been generally appreciated, is that the polarization is a statistical-mechanical phenomenon, and that the Derbenev-Kondratenko formula in fact describes the equilibrium population of electron spins. The formula involves an ensemble average over the distribution of electrons, which is essential to understand quantitatively the phenomenon of "spin resonances," in particular the so-called "nonlinear," or "higher-order," resonances.[6]

For spin-$\frac{1}{2}$ particles, the polarization density matrix is specified completely by a three-component real vector

$$\mathbf{P} \equiv P\hat{\mathbf{P}} = [(N_\uparrow - N_\downarrow)/N_0]\hat{\mathbf{P}}, \tag{1}$$

where $N_{\uparrow,\downarrow}$ denotes the number of electrons with spin projection $\pm\frac{1}{2}$ along the direction $\hat{\mathbf{P}}$, and $N_0 = N_\uparrow + N_\downarrow$ is the total number of electrons, which is constant. We shall calculate the equilibrium values of $N_\uparrow$, $N_\downarrow$, and $\hat{\mathbf{P}}$. If $p_\downarrow$ and $p_\uparrow$ denote the probability per unit time for flipping spin from up to down, and vice versa, then in equilibrium one must have $p_\downarrow N_\uparrow = p_\uparrow N_\downarrow$, whence

$$P = (p_\uparrow - p_\downarrow)/(p_\uparrow + p_\downarrow). \tag{2}$$

These statements do not depend on the chosen axis of quantization, but the description, and calculation, of the equilibrium state of the ensemble is simplified by the use of certain preferred quantization axes. The quantization axis we use is described below: Its use simplifies the determination of the magnitude and direction of the equilibrium polarization. The model we treat assumes that the individual electrons are independent, and that the emission of distinct photons is uncorrelated. In that case $p_\uparrow$ and $p_\downarrow$ are proportional to the number of photons which cause a spin flip along the direction $\hat{\mathbf{P}}$ emitted per unit time; this, in turn, depends on the corresponding power spectra $d\mathcal{P}_{\uparrow,\downarrow}/d\omega$, via

$$p_{\uparrow,\downarrow} \propto \int (d\omega/\hbar\omega)\, d\mathcal{P}_{\uparrow,\downarrow}/d\omega. \tag{3}$$

The calculation entails three steps. First, we describe the solutions of the equations of motion, i.e., the unperturbed trajectories, derived from an appropriate unperturbed Hamiltonian. Next, we calculate the transition probabilities between these trajectories to leading order in perturbation theory, with the aid of (3). Finally, the requirement of statistical equilibrium, together with an ensemble average over the electron distribution, yields the equilibrium degree of polarization, via (2).

The Hamiltonian for the orbital motion is

$$\mathcal{H}_1 = \{[\mathbf{p} - (e/c)\mathbf{A}_{\rm ext}]^2 c^2 + m^2 c^4\}^{1/2} + e\Phi_{\rm ext}, \tag{4}$$

which leads to the Lorentz equation. Here $\mathbf{p}$ is the canonical momentum, while $\Phi_{\rm ext}$ and $\mathbf{A}_{\rm ext}$ are the electromagnetic potentials specified by the design of the accelerator. We shall also need $\boldsymbol{\beta}$, the electron velocity in units of c, $\gamma \equiv (1-|\boldsymbol{\beta}|^2)^{-1/2}$, and the accelerator electromagnetic fields $\mathbf{E}_{\rm ext}$ and $\mathbf{B}_{\rm ext}$. The orbital trajectories consist of oscillations around a central trajectory, the equilibrium closed orbit. It is convenient to use the azimuth θ, instead of time, as the independent variable ($\theta \equiv 2\pi s/L$, where s is the arc length and L is the circumference of accelerator). An orbital oscillation can then be specified by three coordinates $\mathbf{r}(\theta)$ and their canonically conjugate moments $\mathbf{p}(\theta)$, which describe the longitudinal and transverse offsets from the equilibrium closed orbit. These can be expressed[7]

by writing

$$y(\theta)=\begin{pmatrix}\mathbf{r}(\theta)\\ \mathbf{p}(\theta)\end{pmatrix}=\sum_{k=\pm1,\pm2,\pm3}a_kE_k(\theta), \tag{5}$$

where $y=0$ corresponds to the equilibrium closed orbit, $E_k(\theta)$ denotes a normal mode, and the a_k are constants. The normal modes have the property[7]

$$E_k(\theta+2\pi)=e^{i2\pi Q_k}E_k(\theta), \tag{6}$$

where the Q_k are real constants, called the "tunes" of the normal modes. In terms of the a_k and θ, the periodicities of y are described by

$$y(a_ke^{-i2\pi Q_k},\theta+2\pi)=y(a_k,\theta). \tag{7}$$

This will be important to us when discussing the spin trajectories, to which we now turn.

The Hamiltonian for spin motion is[2,8]

$$\mathcal{H}_2=-\frac{e}{mc}\mathbf{s}\cdot\left[\left(a+\frac{1}{\gamma}\right)\mathbf{B}_{\rm ext}-\frac{a\gamma}{\gamma+1}\boldsymbol{\beta}\cdot\mathbf{B}_{\rm ext}\boldsymbol{\beta}-\left(a+\frac{1}{\gamma+1}\right)\boldsymbol{\beta}\times\mathbf{E}_{\rm ext}\right]\equiv\mathbf{s}\cdot\boldsymbol{\Omega}, \tag{8}$$

where $a=(g-2)/2$, in which g is the gyromagnetic ratio. Derbenev and Kondratenko[2] treated only the (unphysical) case $g=2$: We are concerned to check how important are the deviations introduced by nonzero a. In the nonrelativistic limit $\mathcal{H}_2$ reduces to the familiar result $-\boldsymbol{\mu}\cdot\mathbf{B}$, where $\boldsymbol{\mu}=ge\mathbf{s}/2mc$.

From $\mathcal{H}_2$ we can derive the Thomas-BMT (Bargmann-Michel-Telegdi) equation[9] for spin motion $d\mathbf{s}/dt=\boldsymbol{\Omega}\times\mathbf{s}$, where $\boldsymbol{\Omega}$ is evaluated on the orbital trajectory, and so depends on the orbital motion. We cannot therefore adopt the same spin-quantization axis on every orbital trajectory—in general it would not be a solution of the corresponding equation of motion. However, conventional perturbation theory (for calculation of the transition probabilities) demands that the initial and final electron states be solutions of the unperturbed equations of motion, both for orbital and spin motion. For this reason, Derbenev and Kondratenko[2] introduced a vector $\hat{\mathbf{n}}(\mathbf{r},\mathbf{p})$ as the spin-quantization axis. The vector $\hat{\mathbf{n}}$ associated with an orbital oscillation y is defined to be a solution of the Thomas-BMT equation, and may be parametrized by the previous coefficients a_k: $\hat{\mathbf{n}}=\hat{\mathbf{n}}(a_k,\theta)$. Then $\hat{\mathbf{n}}$ displays the same periodicities as y,[2] in (7):

$$\hat{\mathbf{n}}(a_ke^{-i2\pi Q_k},\theta+2\pi)=\hat{\mathbf{n}}(a_k,\theta). \tag{9}$$

The direction of the equilibrium polarization is given by a calculation of the vector $\langle\hat{\mathbf{n}}\rangle_\theta$, the average of $\hat{\mathbf{n}}$ over the coefficients a_k associated with $\hat{\mathbf{n}}$ and y, evaluated at azimuth θ. Because of the parametrization chosen, the computation of this average is easy to visualize—in equilibrium, for a given set of values of $|a_k|$, the particle distribution does not depend on the phases of the oscillations of the particles around the equilibrium closed orbit, and so it is uniformly distributed over all values of the phases of the a_k. It can be shown that this direction is periodic around the ring, i.e.,[6] $\langle\hat{\mathbf{n}}\rangle_{\theta+2\pi}=\langle\hat{\mathbf{n}}\rangle_\theta$, as should be expected.

Next we must consider the relevant matrix elements that describe the transitions of interest. When a photon is emitted, the electron will, due to its energy loss, make a transition to a different orbital trajectory. Let the quantization axes of the initial and final trajectories, at the aximuth θ where the photon is emitted, be $\hat{\mathbf{n}}_i$ and $\hat{\mathbf{n}}_f$, respectively.[10] Then spin flip is defined by a change of spin orientation from $\hat{\mathbf{n}}_i$ to $-\hat{\mathbf{n}}_f$, or vice versa. Thus the relevant matrix elements are $\langle-\hat{\mathbf{n}}_f|\mathcal{H}_{\rm int}|\hat{\mathbf{n}}_i\rangle$ and $\langle\hat{\mathbf{n}}_f|\mathcal{H}_{\rm int}|-\hat{\mathbf{n}}_i\rangle$, where

$$\mathcal{H}_{\rm int}=e(\Phi_{\rm rad}-\boldsymbol{\beta}\cdot\mathbf{A}_{\rm rad})+\mathcal{H}_2(\mathbf{E}_{\rm ext},\mathbf{B}_{\rm ext}\to\mathbf{E}_{\rm rad},\mathbf{B}_{\rm rad}) \tag{10}$$

describes the coupling to the radiation electromagnetic fields. The second term, describing the spin-dependent coupling, has the same form as $\mathcal{H}_2$, but with static fields replaced by radiation fields.

There are also, of course, photon emissions which do not flip spin, described by the matrix elements $\langle\hat{\mathbf{n}}_f|\mathcal{H}_{\rm int}|\hat{\mathbf{n}}_i\rangle$ and $\langle-\hat{\mathbf{n}}_f|\mathcal{H}_{\rm int}|-\hat{\mathbf{n}}_i\rangle$. In addition, there is energy gain in the rf cavities of the accelerator. These processes produce a negligible change in the spin projection along the equilibrium polarization direction $\langle\hat{\mathbf{n}}\rangle_\theta$, but they do have another, important, effect: They lead to the establishment of the equilibrium distribution alluded to above when discussing the ensemble average. It is these nonflip processes which are chiefly responsible for spreading out the particle distribution uniformly over all values of the phases of the coefficients a_k which parametrize the orbital and spin trajectories; thus they determine the direction of the equilibrium polarization, while the spin-flip transitions determine the equilibrium values of the up- and down-spin populations and, hence, the magnitude of polarization.

The time scales involved are very different: The nonflip processes achieve their effect in tens of milliseconds, whereas the spin-flip transitions are much rarer, and require tens to hundreds of minutes to establish the equilibrium degree of polarization. Since the direction of polarization is established so quickly, this leads to a clear distinction between the concepts of spin-flip and non-flip, and of the roles they play—it would otherwise be difficult to define these terms if the direction of $\mathbf{P}$ were evolving towards its equilibri-

um value on a time scale comparable to the buildup time of the equilibrium magnitude of $\mathbf{P}$. Because of this difference in time scales, we are able to easily identify the matrix elements relevant for determining the equilibrium degree of polarization.

Returning to these matrix elements, it is the second term in $\mathcal{H}_{\rm int}$ that is generally regarded as the source of spin-flip synchrotron radiation. Notice, however, that because $\hat{\mathbf{n}}_i$ and $\hat{\mathbf{n}}_f$ are not necessarily parallel, even the spin-independent part of $\mathcal{H}_{\rm int}$ yields a nonzero contribution, which is comparable in practice to that of the spin-dependent term. This is a mechanism not hitherto noticed *per se*. If the photon energy is $\hbar\omega$, and the electron energy is E, and we define $\Delta\gamma = -\hbar\omega/mc^2$, then it is convenient to write

$$\hat{\mathbf{n}}_f \simeq \hat{\mathbf{n}}_i + \Delta\gamma \frac{\partial\hat{\mathbf{n}}}{\partial\gamma} = \hat{\mathbf{n}}_i - \frac{\hbar\omega}{E}\left(\gamma\frac{\partial\hat{\mathbf{n}}}{\partial\gamma}\right), \tag{11}$$

where the derivative $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$ is a measure of the extent to which $\hat{\mathbf{n}}_i$ and $\hat{\mathbf{n}}_f$ are not parallel. This represents the new spin-flip mechanism that leads to terms in $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$ in the Derbenev-Kondratenko formula.

To calculate the power spectra, we follow Schwinger[4] and use the formula

$$\frac{d\mathcal{P}}{d\omega} = \frac{\omega^2}{4\pi^2 c}\mathrm{Re}\left\{\int\left[\frac{1}{c^2}\mathbf{j}(\mathbf{r},t)\cdot\mathbf{j}^*(\mathbf{r}',t') - \rho(\mathbf{r},t)\rho^*(\mathbf{r}',t')\right]e^{-i\omega[t'-t-\hat{\mathbf{k}}\cdot(\mathbf{r}'-\mathbf{r})/c]}d\Omega\,dt'\right\}, \tag{12}$$

where ω is the frequency of the photon, and $\hat{\mathbf{k}}$ its direction of propagation. We get the charge density ρ and the current density $\mathbf{j}$ from the matrix elements of $\mathcal{H}_{\rm int}$. To do so, we substitute for $\mathbf{E}_{\rm rad}$ and $\mathbf{B}_{\rm rad}$ in $\mathcal{H}_{\rm int}$ by writing $\mathbf{E}_{\rm rad} = -(i\omega/c)(\mathbf{A}_{\rm rad} - \hat{\mathbf{k}}\Phi_{\rm rad})$ and $\mathbf{B}_{\rm rad} = -(i\omega/c)\hat{\mathbf{k}}\times\mathbf{A}_{\rm rad}$. We then read off the charge- and current-density operators $\rho_{\rm op}, \mathbf{j}_{\rm op}$ via $\mathcal{H}_{\rm int} = j_\mu A^\mu/c = \rho_{\rm op}\Phi_{\rm rad} - \mathbf{j}_{\rm op}\cdot\mathbf{A}_{\rm rad}/c$. The quantities ρ and $\mathbf{j}$ that appear in (12) are then the appropriate matrix elements of $\rho_{\rm op}$ and $\mathbf{j}_{\rm op}$, e.g., for spin flip from $\hat{\mathbf{n}}_i$ to $-\hat{\mathbf{n}}_f$, we have $\rho = \langle -\hat{\mathbf{n}}_f|\rho_{\rm op}|\hat{\mathbf{n}}_i\rangle$, etc.

The approximations made in the subsequent calculation are the usual ones in the field, and the integrals encountered are the same.[4,5,8] The details of the algebra are lengthy: Some are given in Ref. 3, including expressions for the power spectra $d\mathcal{P}_{\uparrow,\downarrow}/d\omega$. The final expression for the equilibrium degree of polarization is

$$P = \frac{8}{5\sqrt{3}}\frac{\langle\oint(d\theta/|R|^3)\{(1+\frac{14}{3}a)\hat{\mathbf{b}}\cdot\hat{\mathbf{n}} - (1+\frac{3}{2}a)\hat{\mathbf{b}}\cdot\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)\}\rangle}{\langle\oint(d\theta/|R|^3)\{1+\frac{37}{9}a - (\frac{2}{9}+\frac{13}{3}a)(\hat{\mathbf{n}}\cdot\hat{\mathbf{v}})^2 + \frac{11}{18}|\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)|^2\}\rangle}, \tag{13}$$

to first order in $a = (g-2)/2$. Here $\hat{\mathbf{b}}$ is the direction of the local accelerator magnetic field, R is the local radius of curvature of the trajectory, and $\hat{\mathbf{v}}$ is the direction of electron motion. The angular brackets denote an ensemble average over the distribution of particles. Derbenev and Kondratenko[2] obtained the above result for $a = 0$, but in this more detailed form, it seems to be new. Since $a \simeq 10^{-3}$, and the coefficients in (13) depend weakly on a, it is of interest to ask whether the polarization can be strongly influenced by a nonzero value of $g-2$.

To study this, let us begin with the case of circular motion in a uniform, static magnetic field $\mathbf{B}_{\rm ext} = B\hat{\mathbf{z}}$. Then the Lorentz and Thomas-BMT equations reduce to

$$\frac{d\boldsymbol{\beta}}{dt} = \frac{eB}{mc\gamma}\boldsymbol{\beta}\times\hat{\mathbf{z}}, \quad \frac{d\mathbf{s}}{dt} = \frac{eB}{mc\gamma}(\gamma a+1)\mathbf{s}\times\hat{\mathbf{z}}, \tag{14}$$

In this case $\hat{\mathbf{n}}\cdot\hat{\mathbf{b}} = 1$, $\hat{\mathbf{n}}\cdot\hat{\mathbf{v}} = 0$, and $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma) = 0$ everywhere. Then $P = (8/5\sqrt{3})(1+5a/9)$.[11] Neglect of the term in a yields the original result of Sokolov and Ternov.[1] In this idealized model, P is a constant, and does not depend on the details of the accelerator. In more realistic cases, however, the behavior of P is strongly dependent on such details, as manifested by the existence of spin resonances. These resonances are caused by the behavior of $\hat{\mathbf{n}}$ and $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$, which depend *strongly* on the value of a, since by (14), the spin-precession frequency is larger by a factor $\gamma a+1$ than the orbital (Larmor) rotation frequency. Now $\gamma a+1 \simeq 10$–100 in existing or proposed high-energy electron storage rings. The spin thus tends to precess much faster than the velocity when traversing a static magnetic field. In the above example, $\hat{\mathbf{n}} = \hat{\mathbf{z}} = \text{const}$; but in a more realistic model, where $\mathbf{B}_{\rm ext}$ varies, and $\hat{\mathbf{n}}$ must precess around it, it is easy to see that $\hat{\mathbf{n}}$ and $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$ are sensitive functions of a. It is well known to workers in the field, and clear from the above remarks, that the equilibrium direction of the polarization is a strong, explicit function of $g-2$. The important dependence of the equilibrium *magnitude* of the polarization on the magnetic-moment anomaly, however, is *implicit*.

A number of remarks are in order. First, the foregoing justifies the way in which the equilibrium polarization has thus far been calculated in practice[7,12,13] —the Derbenev-Kondratenko expression is used, *but* $\hat{\mathbf{n}}$ and $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$ are calculated with $a \neq 0$. An important goal of these calculations is the analysis of the phenomenon of spin resonances.[7,12,13] Near a resonance, the polarization almost vanishes, and so the accurate determination of resonances is essential to the design of an electron accelerator in which it is desired

to achieve a high degree of polarization. A clear understanding of the roles played by the component parts of the Derbenev-Kondratenko formula, or of (13), in determining the degree of polarization is thus desirable.[6]

Next, (13) is an expression of statistical equilibrium in the transitions between the spin-up and -down populations, and the ensemble average is essential. This aspect of the polarization process has not generally been emphasized.[11] Further, many workers have had difficulty in interpreting the meaning of the quantity $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$. Perhaps the reason for this is the failure to realize that $\hat{\mathbf{n}}$ is not merely a fixed vector, but a vector field—a function of the coefficients a_k. From our analysis, it is clear that $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$ is also a vector field, and has, in general, a different value on distinct orbital trajectories. The usefulness of $\hat{\mathbf{n}}$ itself lies in the ease with which the equilibrium behavior of the electron spin population can be described: The key lies in the parametrization of $\hat{\mathbf{n}}$. By choosing a spin quantization axis whose periodicities are linked to those of the orbital trajectory on which it is defined, we can easily describe the calculation of $\langle\hat{\mathbf{n}}\rangle_\theta$, and also the matrix elements of $\mathcal{H}_{\rm int}$ relevant for determining the equilibrium magnitude of the polarization.

In this context, let us repeat the argument for the origin of polarization in electron storage rings. When an electron emits a photon, its spin sometimes flips. This can be either because of the direct interaction of the spin operator with the photon field in the interaction Hamiltonian, or because the initial- and final-spin quantization axes are not parallel. Polarization develops because the transition probabilities are not equal for flips in opposite directions.

In summary, the above derivation not only shows that the Derbenev-Kondratenko formula may be derived using semiclassical techniques, but also helps to elucidate several aspects of the polarization process not evident from the work of the original authors.

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[10]We are assuming point emission of a photon, which is standard in accelerator physics. In modern machines, the length of formation of the radiation (see Ref. 2) is typically a few centimeters, and $\hat{\mathbf{n}}$ and $\gamma(\partial\hat{\mathbf{n}}/\partial\gamma)$ do not change appreciably in such an interval.

[11]Our result agrees with those in Refs. 2 and 8, where the polarization is calculated to all orders in $g-2$ for this idealized model.

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