Simple System with Quasiperiodic Dynamics: A Spin in a Magnetic Field

Bill Sutherland
Department of Physics, University of Utah, Salt Lake City, Utah 84112
(Received 28 April 1986)

A very simple system with quasiperiodic dynamics is introduced, consisting of a single spin, with spin $\frac{1}{2}$, in a pulsed magnetic field. The pulses are of two types, and the two types alternate in a quasiperiodic way. By adapting renormalization-group and dynamical-systems techniques first introduced in the study of one-dimensional quasiperiodic structures, I characterize the long-time behavior as a function of the experimental parameters. Within different well-defined regions of parameter space, the time correlations decay (a) more slowly than a power law, (b) as a power law, or (c) faster than a power law, and probably exponentially.

PACS numbers: 71.55.Jv, 71.50.+t

There has been much recent interest in quasiperiodic structures in one, two, and three dimensions. These systems are intermediate between the completely periodic perfect crystals and the random or disordered amorphous solids. A major push for the understanding of these structures was given by the experiments of Schechtman *et al.*,[1] which seem to show evidence for a quasicrystal in the material $Al_{0.86}Mn_{0.14}$. At the same time, the one-dimensional quasiperiodic Fibonacci lattices have been studied in much depth, with mathematical techniques of great beauty and power. As I will make frequent use of these results in this Letter, I will now give these references in detail. The first application of dynamical-systems techniques to quasiperiodic structures was in the Letters of Kohmoto, Kadanoff, and Tang[2] and Ostlund *et al.*[3] In particular, Kohmoto, Kadanoff, and Tang introduced a dynamical system equivalent to the Fibonacci series of transfer matrices, a reduced dynamical system of the traces, and found an invariant of this map. Many results were anticipated in an earlier Letter of Kohmoto.[4] A particularly elegant extension of this geometrical understanding was presented by Kohmoto and Oono[5] and Kohmoto.[6] A classification of invariant structures of these maps has been given by Kadanoff.[7]

Although there has been much progress in the theoretical understanding of quasiperiodic structures, and many exotic features have been identified, the experimental realization of such structures is difficult. The exact nature of the alloy $Al_{0.86}Mn_{0.14}$ is still unresolved, while no examples of one-dimensional quasiperiodic lattices have been found in nature. The best prospect for such a one-dimensional lattice is probably to make it. But these experiments are difficult, and although it is theoretically very interesting to vary the parameters of the structure, this usually requires making a new sample.

For these reasons, I here propose a very simple system in which the quasiperiodicity is introduced through the time dependence of the dynamics instead of the structure. Thus it is a very simple matter to repeat the experiment and change the parameters. Yet, as I will show, the consequences of quasiperiodicity—now reflected in the dynamics—are just as interesting as for the quasiperiodic structures. In the course of this investigation, I will extend the theory of the dynamical-systems approach. It is my hope that this Letter might lead to the design of feasible experiments.

To introduce what I suppose is probably the simplest such system, I wish to consider the dynamics of a single spin with spin angular momentum $\hbar/2$, in a time-dependent magnetic field $\mathbf{B}(t)$. Thus the equation of motion is $i\hbar\, d\Psi/dt = -\mu \mathbf{B}(t)\cdot\boldsymbol{\sigma}\Psi$. Here Ψ is the wave function for the spin, a two-component column vector; μ is the magnetic moment of the spin; $\boldsymbol{\sigma}$ are the 2×2 Pauli spin matrices. For the time dependence of the magnetic field, we choose it to be pulsed, so that $\mathbf{B}(t_n)=0$ for times t_n between pulses. Thus the nth pulse occurs between t_{n-1} and t_n. The pulses are only of two types, which we designate A or B.

Suppose that the A pulse occurs between time t_0 and time t_1. This will be produced by a magnetic field of strength $B_A(t)$ in a direction which we take to be constant. Thus the wave function $\Psi(t_1)$ after the pulse is related to the wave function $\Psi(t_0)$ before the pulse through a unitary transformation given by a 2×2 unitary matrix,

$$A=\exp[(i\mu/\hbar)\boldsymbol{\sigma}\cdot\int \mathbf{B}_A(t)\,dt].$$

The integral is taken from t_0 to t_1. We will abbreviate this matrix as

$$A=\exp(i\boldsymbol{\alpha}\cdot\boldsymbol{\sigma})=\cos(\alpha)+i\sin(\alpha)\boldsymbol{\sigma}\cdot\boldsymbol{\alpha}/\alpha.$$

The parameter α is the length of the vector $\boldsymbol{\alpha}$, and is between 0 and π. The direction of $\boldsymbol{\alpha}$ is $\boldsymbol{\alpha}/\alpha$ and can point to anywhere on the surface of a unit sphere.

A similar expression gives us a unitary matrix B for the pulse of type B, and it is parametrized by the vector $\boldsymbol{\beta}$ in a similar manner:

$$B=\exp(i\boldsymbol{\beta}\cdot\boldsymbol{\sigma})=\cos(\beta)+i\sin(\beta)\boldsymbol{\sigma}\cdot\boldsymbol{\beta}/\beta.$$

The dynamics of the spin is now simply a specification of the type of pulses, as they occur. Thus, if we first apply an A pulse, then a B pulse, next another A pulse, $\Psi(t_3)=\Psi_3=ABA\Psi_0$. In general, Ψ_n

$=U(n)\Psi_0$, where $U(n)$ is a 2×2 unitary matrix given as a string of n A's and B's. All finite strings are contained in the infinite work $U(\infty)=U$. All the dynamics after $t=0$ consists in the specification of this infinite string.

The magnetic field $\mathbf{B}_A$ can point in any direction, and so we choose our coordinate system so that $\mathbf{B}_A$ is along the positive z direction. Thus A is simply specified by the parameter α, or equivalently by $y=\mathrm{Tr}(A)/2=\cos(\alpha)$.

For the magnetic field $\mathbf{B}_B$, after we specify the angle Γ that $\mathbf{B}_B$ makes with $\mathbf{B}_A$, the direction is arbitrary, and so we choose our coordinate system so that $\mathbf{B}_B$ is in the x-z plane, pointing in the positive x direction. Finally, we specify the magnitude of B by giving the parameter β, or equivalently by

$$x=\mathrm{Tr}(B)/2=\cos(\beta).$$

We shall call such an orientation for the ordered pair (B,A) the standard alignment. Any other orientation (B',A') can be brought into standard alignment by a rotation, carried out by a rotation matrix R, which itself is in SU(2), through $(B',A')=R(B,A)R^{-1}$.

Thus the experimental arrangement is specified by the three parameters (α,β,Γ), each of which ranges between 0 and π. If we take the trace of the matrix product BA, then

$$z=\mathrm{Tr}(BA)/2$$
$$=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)\cos(\Gamma).$$

The equivalent parameters $\mathbf{r}=(x,y,z)$ will be important in the subsequent discussion.

In this Letter I wish to emphasize the application of this experimental arrangement to the investigation of quasiperiodic systems, a subject of much current interest. In particular, let us consider a dynamics based on the Fibonacci numbers F_k, defined by $F_{k+1}=F_{k-1}+F_k$, $F_0=F_1=1$. These numbers increase exponentially so that $F_k\to\phi^k$ for large k, where $\phi=(1+\sqrt{5})/2$ is the golden mean.

Then we specify the $U_k=U(F_k)$ by the Fibonacci rule

$$U_{k+1}=U_{k-1}U_k,\quad U_1=A,\quad \text{and}\quad U_2=BA.$$

Thus, reading from right to left, the first few pulses in the experiment are

$$\ldots ABABAABABAABAABABAABA.$$

This sequence is obviously not periodic, yet not random, with much self-similarity.

It was the important contribution of Kohmoto, Kadanoff, and Tang to point out that this recursion scheme gives a dynamical system on pairs of matrices. Let $B_k=U_{k-1}$ and $A_k=U_k$; then the recursion scheme is equivalent to the pair of equations $B_{k+1}=A_k$, $A_{k+1}=B_kA_k$, with initial conditions $B_1=B$, $A_1=A$. This is an automonous dynamical system on a six-dimensional vector space of ordered pairs of matrices (B,A), given by the matrix map $(B,A)\to(B',A')=(A,BA)$. Next, following Kohmoto, Kadanoff, and Tang, I show that it contains a simpler dynamical system.

The following result is a simple extension of a result of Kohmoto, Kadanoff, and Tang, and enables us to establish a dynamics on the traces of three consecutive matrices: Let A,B,C,D be matrices in SL(2,C), and $D=AC$, $C=BA$. Then $\mathrm{Tr}D+\mathrm{Tr}B=\mathrm{Tr}C\,\mathrm{Tr}A$. (Tr is the trace of the matrix.)

We make use of the result by considering the transformation $B\to ABA=B'$, $A\to BAABA=A'$, so that $BA\to ABABAABA=B'A'$. We now apply the previous result successively to establish that $\mathrm{Tr}B'=\mathrm{Tr}(BA)\mathrm{Tr}A-\mathrm{Tr}B$, $\mathrm{Tr}A'=\mathrm{Tr}B'\mathrm{Tr}(BA)-\mathrm{Tr}A$, $\mathrm{Tr}(B'A')=\mathrm{Tr}A'\mathrm{Tr}(BA)-\mathrm{Tr}A$. If we treat the traces themselves as independent variables, $x=\mathrm{Tr}B/2$, $y=\mathrm{Tr}A/2$, $z=\mathrm{Tr}(BA)/2$, then the variable z is independent of x,y and we now have a dynamical system on $(x,y,z)=\mathbf{r}$, given by the trace map $\mathbf{r}=(x,y,z)\to\mathbf{r}'=(x',y',z')=(y,z,2yz-x)$.

In terms of the previous U_k matrices, we have $x_k=\mathrm{Tr}\,U_{k-1}/2$, $y_k=\mathrm{Tr}\,U_k/2$, $z_k=\mathrm{Tr}\,U_{k+1}/2$. Now the traces are iterated by the trace map $\mathbf{r}_k\to\mathbf{r}_{k+1}=(y_k,z_k,2y_kz_k-x_k)$, with the initial conditions given by the experimental parameters through $x_1=\cos(\beta)$, $y_1=\cos(\alpha)$, $z_1=\cos(\alpha)\cos(\beta)-\sin(\alpha)\times\sin(\beta)\cos(\Gamma)$.

Further, as pointed out by Kohmoto, Kadanoff, and Tang, the following quantity is an invariant of this dynamical system:

$$I=x^2+y^2+z^2-2xyz-1$$
$$=-[\sin(\alpha)\sin(\beta)\sin(\Gamma)]^2.$$

The invariance is easy to verify by substitution. The consequence is that the trace system is seen to move on a two-dimensional manifold. The second expression gives the range of I as $-1\le I\le 0$. If $I=0$, then A,B commute, and all $U(n)$ are elements of an Abelian subgroup of SU(2). On the other hand, if $I=-1$, A,B generate the quaternion subgroup of SU(2), a finite group consisting of the eight elements $\pm I$, $\pm i\sigma_j$.

Suppose that A and B are elements of a finite subgroup of SU(2). Then clearly all $U(n)$ are elements of the same subgroup. Finally, since the subgroup is finite the phase space of the dynamical system is finite and hence all points must be K-cycles of the dynamical map.

For A,B elements of a finite subgroup, the invariant I can take only discrete values. In fact, however, we find fixed points of the trace map as curves intersecting the invariant surfaces. Suppose we find a K-cycle of the trace map. What are the implications for the matrix map?

Since the traces of three consecutive matrices repeat, this means that the parameters of two consecutive U's, say $B_{k+K}=U_{k+K-1}$ and $A_{k+K}=U_{k+K}$, repeat, as does the angle between. Thus, this pair (B_{k+K},A_{k+K}) is rigidly rotated, and therefore the K-times iterated pairs are equivalent, up to a rotation R, to the original pair (B_k,A_k). In terms of the matrix map, we have $(B_{k+K},A_{k+K})=R(B_k,A_k)R^{-1}$. I emphasize that R is the same for all k.

Therefore, an invariant set for the K-times iterated matrix map is the set of all distinct pairs of matrices $R^n(B_k,A_k)R^{-n}$, for all integer n. This may be finite or infinite, depending on whether R is rational or not. Different invariant sets are further labeled by the index k from 1 to K. If the cycles of the trace map correspond to a finite group, then we know that the invariant set of the matrix map is a set of points. Thus R must be rational. If R is irrational, on the other hand, then the invariant set consists of K one-parameter submanifolds.

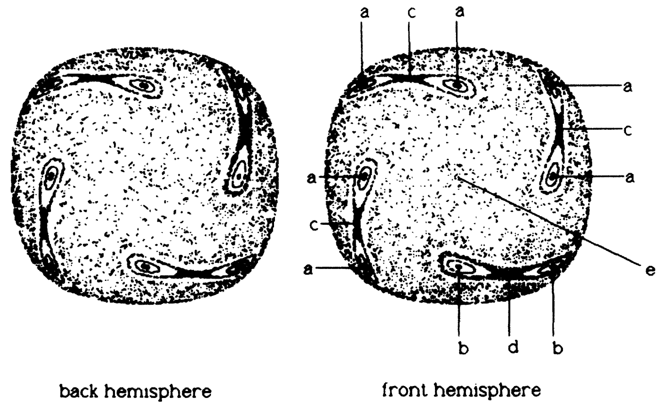

FIG. 1. The two hemispheres of the invariant surface of the trace map with $I=-\frac{1}{2}$ are shown. All points outside the figure eights are points on a single orbit. The letters designate fixed points of the iterated trace map as follows: (a) elliptic twelve-cycle, (b) elliptic four-cycle, (c) hyperbolic six-cycle, (d) hyperbolic two-cycle, and (e) hyperbolic six-cycle.

How is R determined? Take an initial matrix pair $(B(\mathbf{r}),A(\mathbf{r}))$ in standard alignment with parameter $\mathbf{r}$. Iterate once to find $(B',A')=(A,BA)$, with parameter $\mathbf{r}'$. Then (B',A') can be brought into standard alignment by a rotation $S(\mathbf{r})$, to give $(B',A')=S(\mathbf{r})\times(B(\mathbf{r}'),A(\mathbf{r}'))S(\mathbf{r})^{-1}$. When we repeat this process k times, we find the expression

$$(B_{k+1},A_{k+1})=S(\mathbf{r}_1)\cdots S(\mathbf{r}_k)(B(\mathbf{r}_{k+1}),A(\mathbf{r}_{k+1}))S(\mathbf{r}_k)^{-1}\cdots S(\mathbf{r}_1)^{-1}.$$

Thus for a K-cycle of the trace map we have $\mathbf{r}_{K+1}=\mathbf{r}_1$, so that we can make the identification $R=S(\mathbf{r}_1)\cdots S(\mathbf{r}_k)$. This expression must be invariant set if we translate along the orbit of the trace map. Similarly, if the invariant set of the trace map is a collection of closed curves, as occurs about an elliptic or stable K-cycle—eigenvalues with absolute value 1—the invariant set of the matrix map is a set of tori. Thus, in this sense the dynamics of the rotations is trivial or integrable; the trace dynamics determines all.

In Fig. 1 I show typical orbits of the trace map on one hemisphere of the invariant surface, which has the topology of a sphere, for $I=-\frac{1}{2}$. Note that all points except for the closed curves inside the figure eights are points on a single orbit. The other hemisphere looks the same up to a rotation. We identify an ellptic twelve-cycle, an elliptic four-cycle, a hyperbolic six-cycle, and a hyperbolic two-cycle. Another hyperbolic six-cycle is contained within the large chaotic orbit.

Although there is certainly much other structure in the trace map, these shorter cycles dominate the gross features of the trace map at all values of I. The six-cycle $(0,y,0,0,-y,0)$ is hyperbolic at all $I=y^2-1$. The two-cycle (x,y) and six-cycle $(x,-y,-x,y,-x,-y)$ with $x+y=2xy$ and $xy<0$ are a pair which together have the tetrahedral symmetry of the invariant surface. I is given by $I=-\frac{9}{8}+(x-\frac{1}{4})^2+(y-\frac{1}{4})^2$; they exist for all I and are elliptic for $I<-\frac{9}{16}$. At $I=-\frac{9}{16}$ they become hyperbolic, and bifurcate to produce an additional elliptic four-cycle $(x,-\frac{1}{2},\frac{1}{2}-x,-\frac{1}{2})$ and twelve-cycle $(x,\frac{1}{2},x-\frac{1}{2},-\frac{1}{2},-x,\frac{1}{2},\frac{1}{2}-x,\frac{1}{2},-x,-\frac{1}{2},x-\frac{1}{2},\frac{1}{2})$, respectively, with $I=-\frac{9}{16}+(x-\frac{1}{4})^2>-\frac{9}{16}$.

In Fig. 2 I show the direction of the matrices A_k in the matrix sequence, by points on the unit sphere, near the two-cycle, for a value of the invariant I for which the two-cycle has become stable. Note that since this is a projection of the phase space of the dynamical map, winding around the torus is reflected in the braid structure of Fig. 2.

The discrete "time" of the matrix dynamics is the index k of the Fibonacci numbers $F_k\to\phi^k$, while the Fibonacci numbers themselves are the discrete time of

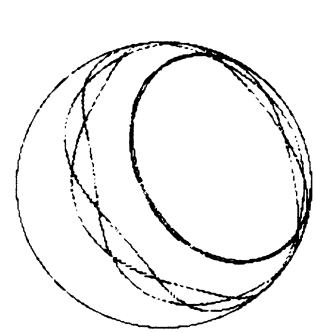

FIG. 2. The direction of the evolution matrix U_k along an orbit of the matrix map, near the elliptic two-cycle of the trace map. The invariant surfaces of the matrix map are two tori, and the braid structure in the figure is a projection of the twists on the tori.

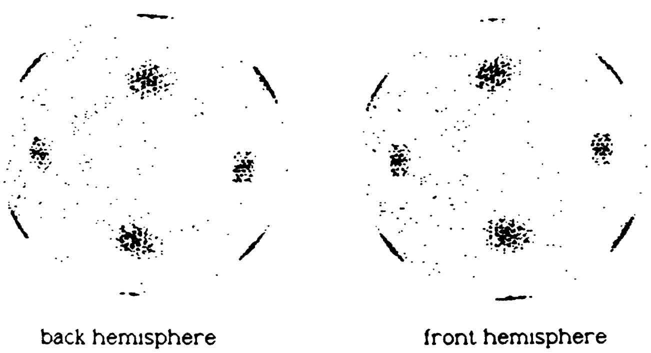

FIG. 3. The direction of the evolution matrix $U(n)$ in time, for an orbit of the trace map near the elliptic twelve-cycle. Note the localization near the directions corresponding to the tetrahedral subgroup, and the further structure in these spots.

the spin system, given by the index n on the wave functions. Thus $n \rightarrow \phi^k$. In Fig. 3 I show the direction of the evolution matrices $U(n)$ after successive pulses, by points on the unit sphere, near the elliptic twelve-cycle, for a value of the invariant I near $-\frac{1}{2}$, the value of I shown in Fig. 1. Note the localization about discrete directions. The traces of $U(n)$ are likewise localized near the values $0, \pm\frac{1}{2}, \pm 1$; the first three are from the twelve-cycle, while the last two are from $\pm$ the identity.

We can understand these results as follows. Near a stable elliptic cycle of the trace map, the traces and matrices rotate with constant angular velocities in the Fibonacci index k, and thus slow down logarithmically in the time index $n \rightarrow \phi^k$; essentially they stop and are localized. Thus, near a stable elliptic cycle of the trace map, the autocorrelation functions decay more slowly than a power of the time.

Similarly, near a hyperbolic cycle of the trace map, the autocorrelation functions decay as a power of the time, with the power being given by the Lyapunov exponent of the trace map. Finally, if in the chaotic region of the trace map the escape is faster than exponential in the Fibonacci index, as suggested by the work of Kadanoff and Tang,[8] and instead goes as the Fibonacci number itself, then the autocorrelation functions will decay faster than a power and possibly exponentially in time.

I have thus determined the three regions of parameter space characterized by the qualitative difference of the decay of the time correlations: more slowly than a power, as a power, and faster than a power. In addition, when we are near a stable cycle of the trace map, so that the decay of the time correlations is slower than a power, the time evolution has a self-similar or fractal character, which can be calculated exactly by methods introduced by Sutherland.[9]

Finally, I wish to emphasize that in the calculation I have selected conditions which make the theory particularly tractable. This may not be the optimum choice for the experiment. However, it is reasonable to expect that the qualitative results would be the same for other realizations, such as (1) arbitrary—or even classical—spins, (2) weak relaxation effects, and (3) oscillatory instead of pulsed magnetic fields, as long as the two frequencies are strongly irrational, such as in the golden ratio. The region of parameter space for which the motion is localized can be quite large, the arguments are robust, and the systems are easy to simulate numerically.

I would like to thank Mahito Kohmoto for many helpful discussions.

[1]D. Schechtman, I. Blech, D. Gratias, and J. W. Cahn, Phys. Rev. Lett. **53**, 1951 (1984).

[2]M. Kohmoto, L. P. Kadanoff, and C. Tang, Phys. Rev. Lett. **50**, 1870 (1983).

[3]S. Ostlund, R. Pandit, D. Rand, H. J. Schellnhuber, and E. D. Siggia, Phys. Rev. Lett. **50**, 1873 (1983).

[4]M. Kohmoto, Phys. Rev. Lett. **51**, 1198 (1983).

[5]M. Kohmoto and Y. Oono, Phys. Lett. **102A**, 145 (1984).

[6]M. Kohmoto, in *Chaos and Statistical Methods*, edited by Y. Kuramoto (Springer, Berlin, 1984).

[7]L. P. Kadanoff, to be published.

[8]L. P. Kadanoff and C. Tang, Proc. Natl. Acad. Sci. U.S.A. **81**, 1276 (1984).

[9]B. Sutherland, Phys. Rev. B (to be published).