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Fractal Dimension of Cantori

Wentian Li
Brookhaven National Laboratory, Upton, New York 11973, and Columbia University,[a] *New York, New York 10027*

and

Per Bak
Brookhaven National Laboratory, Upton, New York 11973
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At a critical point the golden-mean Kolmogorov-Arnol'd-Moser trajectory of Chirikov's standard map breaks up into a fractal orbit called a cantorus. The transition describes a pinning of the incommensurate phase of the Frenkel-Kontorowa model. We find that the fractal dimension of the cantorus is $D=0$ and that the transition from the Kolmogorov-Arnol'd-Moser trajectory with dimension $D=1$ to the cantorus is governed by an exponent $\bar{\nu}=0.98...$ and a universal scaling function. It is argued that the exponent is equal to that of the Lyapunov exponent.

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Chirikov's standard map[1] describes the behavior of a Hamiltonian dynamical system which is subjected to a periodic force. The map has periodic orbits, smooth quasiperiodic Kolmogorov-Arnol'd-Moser (KAM) orbits,[2] and chaotic orbits. As the interaction increases the KAM trajectories eventually break up into fractal objects called cantori. The cantori are imbedded in the chaotic regime so that they are mathematically unstable. Nevertheless, it has been convincingly argued that the cantori play an important role for transport properties (Arnol'd diffusion).[3,4]

The orbits of the standard map are also of importance to an entirely different physical problem, namely the static Frenkel-Kontorowa (FK) model of modulated structures. The FK model may describe, for instance a monolayer of rare-gas molecules adsorbed on a periodic substrate.[5,6] The KAM trajectories correspond to free-sliding incommensurate states and the cantori describe pinned incommensurate states, where the particles occupy only a subset of positions relative to a periodic potential. Both the KAM orbits and the cantori form physically stable ground states for the model, and the breakup of the cantorus describes a pinning transition of the incommensurate structure.

We have studied the metric properties of the cantorus with golden-mean winding number (or wave vector for the FK model) near the transition. Surprisingly, the measure of the cantorus goes immediately to zero at the transition. The small gaps emerging in the KAM surface thus conspire to fill up the surface completely, except for a Cantor set. The fractal dimension of this Cantor set is zero, and so the cantorus is extremely thin. The crossover from the KAM trajectory of measure 1 to the cantorus is characterized by an exponent $\bar{\nu}=0.98....$ We argue that this exponent is identical to the exponent for the Lyapunov exponent. A similar direct connection between the Lyapunov exponent and the fractal dimension has been conjectured for strange attractors in dissipative turbulent systems.[7] Furthermore, we have evidence that the transition is governed by a crossover scaling function.

The Frenkel-Kontorowa model consists of an infinite array of atoms, interacting by harmonic springs, in a periodic potential:

$$H=\sum \tfrac{1}{2}(x_i-x_{i-1}-\delta)^2 + [k/(2\pi)^2](1-\cos 2\pi x_i), \quad (1)$$

where x_i is the position of the ith atom and δ is a misfit parameter.

The extremal configurations are found by differentiating (1) with respect to x_i:

$$x_{n+1} = x_n + r_{n+1},$$
$$r_{n+1} = r_n + (k/2\pi)\sin(2\pi x_n). \quad (2)$$

These equations are the recurrence relations for the standard map. Orbits of the map are characterized by their winding number $\omega = \lim(x_i - x_0)/i$, $i \to \infty$. For small enough k there exist smooth invariant incommensurate 1D orbits, the KAM trajectories predicted by the famous KAM theorem.[2] The orbits are filled up ergodically as the iteration proceeds, and they describe configurations where the entire chain can be displaced continuously with no energy barriers. At $k = k_c = 0.971\,635\ldots$ the last KAM surface with winding number equal to the golden mean $\tau = (\sqrt{5}-1)/2$ breaks up and becomes a Cantor-set-like object, the cantorus (Fig. 1). The corresponding configurations are pinned incommensurate states where the individual particles move discontinuously to overcome energy barriers as the chain is displaced. Although this transition has been studied before[3–6, 8–10] (in particular with respect to the transport properties), and it is generally believed that the cantorus is a Cantor set with fractal dimension between 0 and 1, this point has not been examined carefully.

In order to perform numerical calculations, the golden-mean ratio is approximated in the usual way by a converging series of rational numbers $p/q = p_n/p_{n+1}$, where p_n is the nth Fibonacci number. Because of the duality of the problem as either a mapping problem or a minimization problem the orbits can be found by essentially two different methods. The first is to follow the line of Aubry,[5] looking for orbits by minimizing the energy of atoms in the Frenkel-Kontorowa model. This method turns out to be applicable only

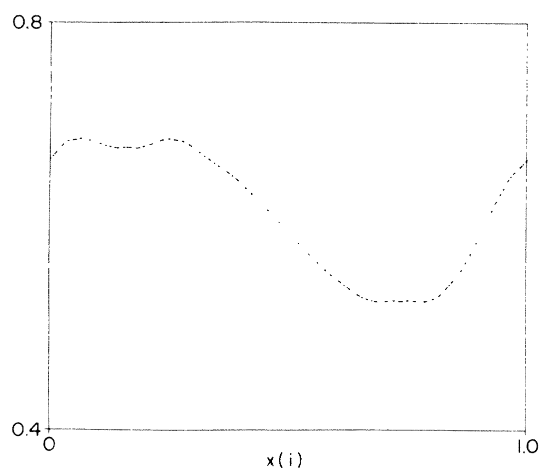

FIG. 1. Cantorus calculated for $\epsilon = k - k_c = 0.02$ and $p = 1597$.

for small system size, for instance $n < p_{15} = 987$, and for the extreme chaotic regime $k \gg k_c$ which we are not interested in here. The second method is to search for periodic solutions directly by a mapping method, where usually a Newton iteration method is used.[11] The two obstacles in using direct mapping with Newton iteration are as follows: (i) There are many "illegal" orbits with the same winding number born out of tangent bifurcations, and (ii) there is extreme sensitivity to initial conditions (because we are in the chaotic regime!) which makes it impossible to stay for many iterations on the orbit with our limited precision.

The new method which we use to overcome these problems is a modified iteration method. The orbit that we are looking for is ordered in the sense that the ith atom is in the jth potential well where $j = \mathrm{int}(iq/p)$. Now, fix $x_1 = 0$ (favorable for p odd) and vary x_2. If x_2 is too large, then one or more particles x_i with $i > 2$ will move out at the right edge of the proper potential well. On the other hand, if x_2 is too small then at least one particle will move out at the left edge. By successive iterations, starting from the first particle in the improper well and continuing to inspect the succeeding particles, we adjust x_2 such that all particles are in the proper well. There is one and only one configuration for which this is the case, which we then locate precisely by a bisection method. A Newton iteration method would also work as the final step. This method has considerable numerical simplification compared with the energy minimization method: The computing time of the latter is proportional to at least p^2, but that of the former is proportional only approximately to p. We have succeeded in locating cantori with p up to 17 711 for $k - k_c = 0.0006$.[12] Figure 1 shows an example of a cantorus, with $k - k_c = 0.02$ as approximated by $p = 1597$.

At the critical point, narrow gaps open up in the KAM surface. The largest gap indicates a regime near the maximum of the potential where there are no particles. The largest gap has been found to be largely responsible for the transport properties of the map[3,4] and increases as $(k - k_c)^{0.721}$. One might, therefore, naively suspect that near k_c the gaps fill up only a vanishingly small part of the KAM surface so that the measure of the remaining cantorus would remain finite. We shall see that it is not so!

In order to study the scaling properties of the cantori we first estimate the measure $m(r)$ "left over" when gaps larger than a certain scale r are removed from the set, following a procedure suggested by Jensen, Bak, and Bohr.[13] In order to locate the gaps we simply start with the largest gap as estimated by the two consecutive points x_i and $x_{i'}$ separated by the largest distance and iterate backwards to locate the smaller gaps.

Figure 2 shows a logarithmic plot of $m(r)/r$ vs $1/r$ for various values of $k - k_c$. The asymptotic slope is

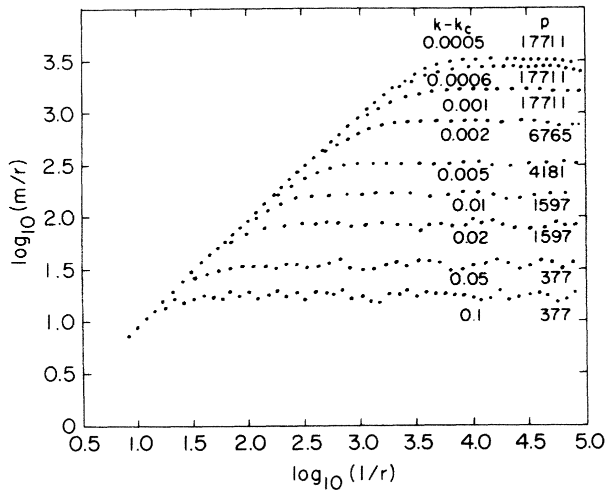

FIG. 2. Measure m of the set in units of the scale r vs $1/r$ for various values of ϵ.

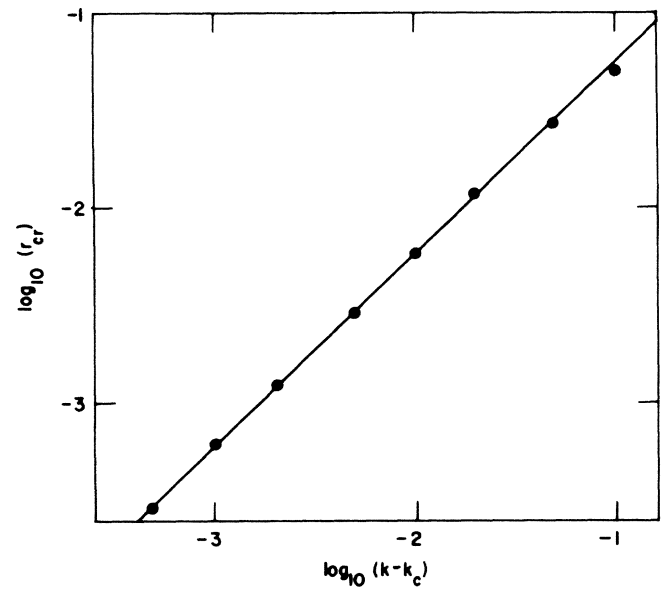

FIG. 3. Crossover scale r_{cr} plotted vs ϵ. The linear behavior indicates an exponent $\bar{\nu} = 0.98 \pm 0.01$. Within numerical accuracy the exponent is identical to ν, the exponent for the Lyapunov exponent.

the fractal dimension D of the orbit. For scales that are not too small, the curve is approximately a straight line with slope $D' = 1$, the dimension of the KAM trajectory. *For small scales there is a crossover to a horizontal line, i.e., the dimensional of the cantorus is $D = 0 \pm 0.02$!* The fact that the slope is less than 1 means that the gaps, however tiny they may be, eventually conspire to fill up the whole interval as soon as k exceeds k_c. The remaining Cantor set is remarkably thin, indicating a strong clustering of the points. If the curves are continued to even smaller scales (not shown in the figure) the gaps estimated above are the true gaps but are increased as a result of finite-size effects, causing the curves to turn steeply downward.[14] When the scale is smaller than the smallest gap found for a given p the measure of course becomes strictly zero.

The crossover points r_{cr}, as estimated from the curves in Fig. 2, have been plotted versus $\epsilon = k - k_c$ (Fig. 3). The slope of the resulting straight line indicates a crossover exponent $\bar{\nu} = 0.98 \pm 0.005$. This exponent appears to be identical to the exponent ν for the Lyapunov exponent for the cantorus.[3] Below we shall present arguments supporting this simple connection.

The existence of a well-defined crossover exponent ν indicates the possibility of a universal crossover scaling function. A proper scaling *Ansatz* could be $m(r)/r = (1/r)^{D'} h(r/\epsilon^{\nu})$, or, since dimension D' of the KAM surface is $D' = 0$

$$m(r) = h(r/\epsilon^{\nu}). \qquad (3)$$

Figure 4 shows $m(r)$ plotted versus r/ϵ^{ν} for various $\epsilon = k - k_c$. Indeed, the points appear to approach a unique curve for small ϵ. The reason that the points for large ϵ do not lie on the curve is obvious: For these values the scale in the crossover region becomes comparable to the largest gap and the measure is strictly 1. For smaller $k - k_c$ this is not a problem because the exponent for the largest gap is smaller than the exponent for the crossover and the scale of the largest gap shifts to the right in Fig. 4 away from the crossover region. For small values of the argument, $h(x) \approx x^{1-D} \approx x$ since $D \approx 0$. For large values of x the function approaches 1, the measure of the KAM trajectory.

So why is the dimension zero and why is the cross-

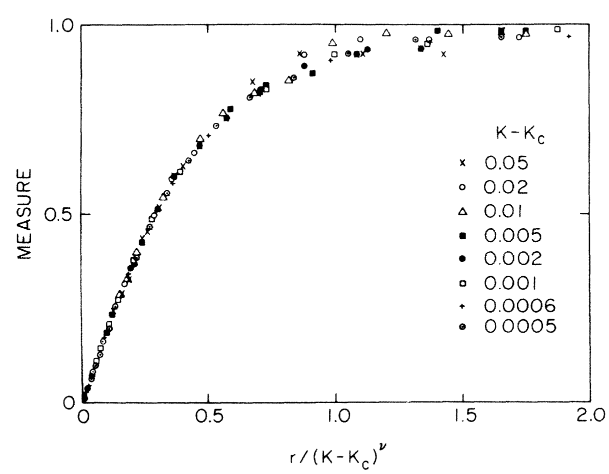

FIG. 4. Measure of the Cantor set plotted vs r/ϵ^{ν}. The points for small ϵ appear to approach a unique scaling function.

over exponent identical to that of the Lyapunov exponent? The following arguments may be useful in understanding the connection. Consider the gap structure for the nth approximation to τ. The p_n gaps obviously fill up everything. Had we instead considered the $(n+1)$th approximation we would have found $n-1$ more points. The new gaps arising can be thought of as arising from the original gaps through $n-1$ backwards iterations, each reducing the gap by a factor $\approx \exp(-\lambda)$ according to the definition of the Lyapunov exponent λ. The new points (and the complete set) are within a distance of $\approx \exp(-p_{n-1}\lambda)$ from the original points. Hence, p_n boxes of size $\exp(-p_{n-1}\lambda)$ would cover the set and the fractal dimension is

$$D = \lim[\ln\{\exp(-p_{n-1})\}/\ln(p_n)] = 0.$$

Since the size of the xth largest gap is roughly $\lambda \exp(-x\lambda)$, the "crossover" number of gaps n needed to fill up a certain fraction f of the interval is obviously proportional to $1/\lambda$. The crossover scale $r_{\rm cr}$ is the size of this nth gap, i.e., $r_{\rm cr} \approx \lambda \approx \epsilon^{\nu}$, according to the definition[3] of the exponent ν and in agreement with our numerical results.

In conclusion, we have found by means of numerical and analytical considerations the rather counterintuitive result that the KAM trajectory immediately dissolves into a thin Cantor set of dimension $D=0$ at the critical point. The cantori above the critical point are characterized by a crossover scale increasing as the Lyapunov exponent, and we have evidence that the crossover is charaterized by a universal scaling function. These results can be directly translated into statements on the scaling of the set formed by the positions of particles in the FK model.

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[a]Present address.

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[12]When $k-k_c$ is large our method cannot locate long orbits because of the large Lyapunov exponent and consequent sensitivity to the initial point. But then the cantorus is so thin that we do not need many points to reach the crossover region that we are interested in. On the other hand, when $k-k_c$ is very small we can easily generate even longer cycles.

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[14]Note that the present study is different than previous ones in that we are not interested in "finite-size scaling" as the golden mean is approximated by successive p/q. We simply choose one large enough p for each $k-k_c$ $[p \gg (k-k_c)^{-\nu}, \nu = 0.98\ldots]$ to assure that finite-size effects are not important.