Instability of a Thermal Stokes Layer

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We examine experimentally the stability of a Stokes layer in a fluid near a boundary whose temperature is modulated as $T_0 \cos\omega t$. We define an appropriate Rayleigh number for the problem and determine its critical value. Increased stabilization is observed to accompany a reduction in the Prandtl number. We observe hysteresis effects near the critical Rayleigh number, including a double hysteresis loop, which appear qualitatively similar to recent predictions of Roppo, Davis, and Rosenblat.

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Externally modulated hydrodynamic systems are a relatively recent development in studies of hydrodynamic stability and the transition to turbulence.[1-8] Comparison of experiment to theory for slowly modulated Taylor-Couette flow has not been notably successful,[2,9] whereas when modulation is rather rapid, theory and experiment are in quite satisfactory accord.[7,10] Slowly modulated Bénard convection is likewise a difficult problem,[4-6] whereas rapid modulation appears to be, experimentally, a relatively simple and convenient system to study, at least at cryogenic temperatures. In particular, the important time constants of the problem are short (small layer thickness, rapid modulation), and the effects of side walls can be minimized (large effective aspect ratio—see below).

This Letter describes a novel type of thermal-convection experiment which we believe has not been investigated in detail in the past. The principle of the experiment is illustrated in Fig. 1. Figure 1(a) shows the thermal circuit and Fig. 1(b) the electrical circuit. We have a cylindrical cell operating at cryogenic temperatures with helium I as the working fluid. The cell, of radius 0.809 cm, has a stainless-steel walls of thickness 2.54×10^{-2} cm, and top and bottom copper plates whose separation is $d=0.4115$ cm.

The principle of the experiment, suggested to us by Philip Hall, is that one plate is held at a reference temperature T and the other at a temperature $T+T_0\cos\omega t$, where ω is the circular frequency and t the time. The effect of this periodic heating and cooling extends into the fluid a characteristic distance $d_s=(2\kappa/\omega)^{1/2}$, which we shall call the Stokes layer.

We have chosen to modulate the upper plate of our apparatus, whose mean temperature is set at T_u, and to monitor the mean temperature difference between the upper and lower plates $\Delta T = T_L - T_u$ as a function of the modulation amplitude. We define a Rayleigh number for this problem as $N_{\rm Ray}=g\alpha T_0 d_s^3/\kappa\nu$, where g is the acceleration of gravity, and α, κ, and ν are respectively the thermal expansion coefficient, thermal diffusivity, and kinematic viscosity of the fluid, whose Prandtl number is defined as $\sigma=\nu/\kappa$. The thermal circuit for the experiment may be appreciated from Fig. 1(a). The top plate is heated periodically through heater H_c; the mean temperature T_u is set by a heat leak $R_{\rm HL}$ to an external, colder, temperature bath. If the liquid helium is not in motion, the thermal resistance of the liquid, $R_{\rm He}$, gives a linear response. The effect of modulation of the upper plate is then completely reversible, and $\Delta T=0$. On the other hand, if convection begins in a Stokes layer of thickness $d_s \ll d$, the thermal resistance $R_{\rm He}$ is nonlinear, and heat is given up irreversibly by the underlying fluid. This cools the lower plate and gives rise to a temperature difference $\Delta T<0$ via a rectification effect. The

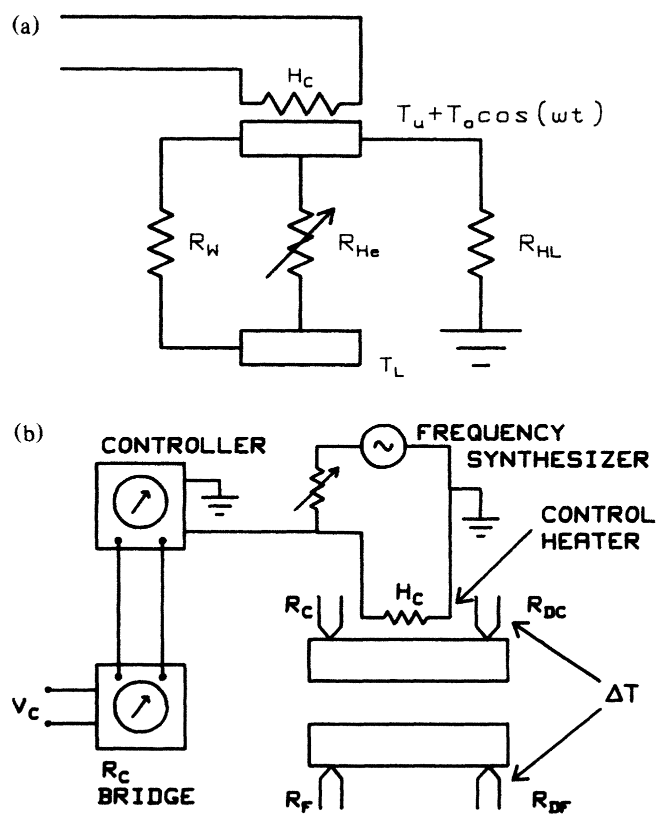

FIG. 1. Schematic diagram of (a) thermal circuit and (b) electrical circuit for the Stokes-layer experiment.

cell walls constitute a parallel thermal resistance R_w back to the upper plate.

The electrical circuit is shown in Fig. 1(b). Under steady conditions, the top plate is maintained at T_u by means of a resistance bridge and temperature controller sensing the germanium resistance R_c and supplying controlled heat to H_c. Thermal modulation is provided by a simple addition of a frequency synthesizer and adjustable series resistance, which superimposes a modulation of the electric current supplied to H_c at a frequency ω. This corresponds to a temperature modulation of amplitude T_0 as determined from measurement of R_c. The temperature difference ΔT is measured by a ratio transformer bridge between matched resistors $R_{\rm DC}$ and $R_{\rm DF}$. The thermal time constants are such that T_L responds slowly to conditions in the unstable layer.

The convective onset is observed experimentally by plotting ΔT, the measure of convective contributions to the heat flux, against the Rayleigh number as the latter is increased in value. Ramping of the Rayleigh number is accomplished by increasing the peak amplitude of the sinusoidal current output to H_c in steps small compared with that found necessary to initiate convective flow. T_0, and hence $N_{\rm Ray}$, are linearly dependent upon the amplitude of the modulated electric current, so that any ramping rate can be applied uniformly, as desired, throughout a run. A waiting time of at least one vertical thermal diffusion time of the entire cell is employed after each step in T_0 and prior to data acquisition.

The temperature difference ΔT is obtained from out-of-balance voltage measurements of the ratio transformer bridge; averages are computed for measurement times of $t=2\pi n/\omega$, n an interger, from which the temperature difference is then recovered through a previous calibration. T_0 is obtained from the resistance bridge measuring R_c by a similar procedure. The output of this bridge, labeled V_c in Fig. 1(b), provides a dc voltage proportional to its off-balance signal. This, in turn, is proportional to the deviation of the top-plate temperature from its mean value T_u. The amplitude of the temperature modulation T_0 is obtained from measurements of this voltage, averaged over integral multiples of the modulation period, using known values of dV_c/dR_c and dR_c/dT. With T_0, the Rayleigh number is computed from values of the fluid parameters evaluated at the temperature T_u and obtained from the cubic spline fits of Barenghi, Lucas, and Donnelly.[11] During an experimental run care is taken to stay well within the linear range of the off balance of the bridge. As an example of the linearity of the modulation, higher harmonics of the temperature difference across the cell in the subcritical (conduction) region were typically measured to be at least 60 dB lower than the amplitude of the fundamental modulation frequency.

Figure 2 is an example of a ΔT-$N_{\rm Ray}$ plot and shows combined results for three runs with frequencies of 0.032, 0.048, and 0.08 Hz and Prandtl number $\sigma=0.49$. Here ΔT is normalized by T_{0c} which is the amplitude of modulation at the convective onset. The frequencies chosen correspond to Stokes-layer thicknesses varying between 10% and 20% of the cell height d. The critical Rayleigh number for these runs is found to have a value of approximately 122, independent of the period of modulation. One of the more interesting features of Fig. 2 is that the onset of convection is "abrupt," i.e., the flow develops at finite amplitude. This is indicative of an inverted bifurcation, and is in contrast with what one expects in the more conventional case of an unmodulated, stationary fluid layer similarly satisfying the Boussinesq approximation.

At the time these results were first obtained, we had a number of interesting conversations with Steve Davis, who suggested that we look for hysteresis about the convective onset. Hysteresis is an obvious manifestation of the bifurcation structure predicted by Roppo, Davis, and Rosenblat[12] for a fluid layer with one, or both, horizontal boundaries subject to a sinusoidal modulation of the temperature. In particular, the theory predicts a finite-amplitude convective onset to a hexagonal planform as the Rayleigh number $N_{\rm Ray}$ is increased past its critical value. Dependence upon initial conditions is expected to give rise to a double hysteresis loop near the critical Rayleigh number, where the first loop (in the direction of increasing Rayleigh number) traverses the regimes of stable conduction and subcritical hexagons, while the second is entirely supercritical and involves a history-dependent transi-

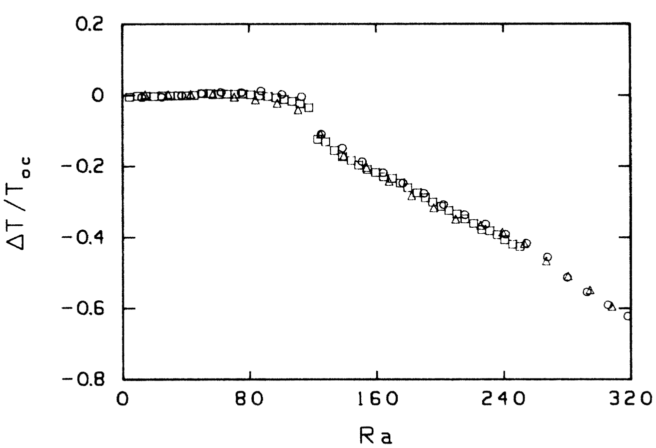

FIG. 2. Mean temperature difference vs Rayleigh number for $\sigma=0.49$ ($T=2.63$ K) and modulation frequencies of 0.032, 0.048, and 0.08 Hz corresponding to circles, squares, and triangles, respectively, as the Rayleigh number is increased. ΔT is normalized by T_{0c}, which is the amplitude of modulation at the convective onset.

tion between hexagons and roll-type convection. This bifurcation picture is essentially the same as that predicted earlier for a stationary convecting fluid layer subject to non-Boussinesq conditions.[13]

Subsequently, we observed a very clear and striking hysteresis effect composed of two loops, which is displayed in Fig. 3(a). A schematic representation of the hysteresis is presented in Fig. 3(b), where bifurcation curves taken to represent two different convection planforms have been drawn to correspond to the data in Fig. 3(a). Specifically, the unhatched portions of the curves were plotted to overlie the actual data points and, together with the hatched portions, to serve as a guide to the eye for purposes of comparison with relevant theoretical plots. In fact, there is remarkable qualitative similarity between Fig. 3(b) and

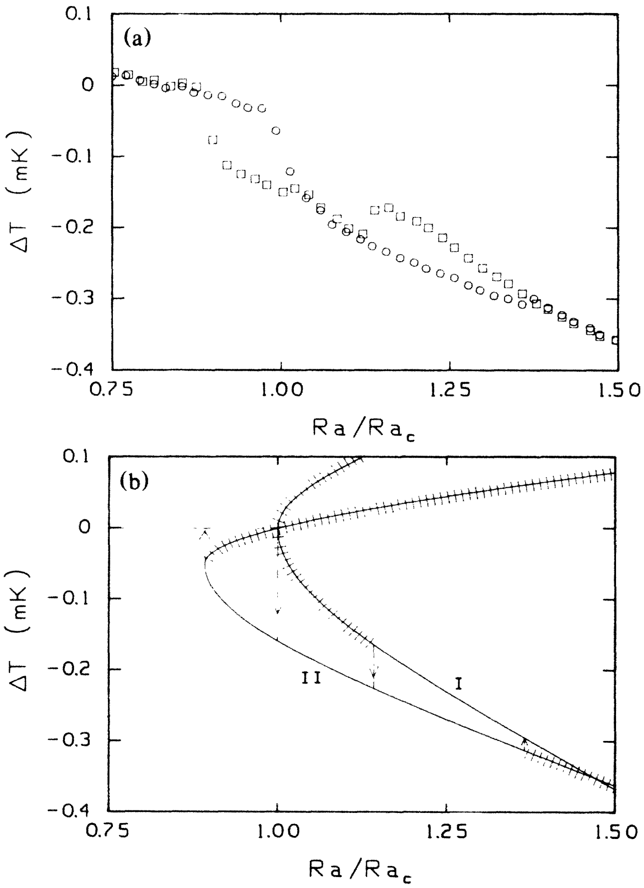

FIG. 3. Results of hysteresis run for $\sigma = 0.49$ and $\nu = 0.048$ Hz. (a) The data corresponding to increasing (circles) and decreasing (squares) Rayleigh numbers. The Rayleigh numbers are normalized by $N_{\mathrm{Ray}\,c} = 122$. (b) A more schematic picture of the experimental results. Hatch marks indicate unstable branches for two convection planforms I and II. The dashed line corresponds to the conduction state and vertical lines with arrows indicate points of transition from one mode to another. Transitions are generally made only in the direction of the arrows as the Rayleigh number is increased or decreased.

the bifurcation diagram presented in Ref. 12 (their Fig. 3).

We note that the dashed horizontal line $\Delta T = 0$ in Fig. 3(b), corresponding to the conduction state of the fluid, is not actually realized experimentally, as is clear from comparison to Fig. 3(a) (see also Fig. 2). Instead, values of ΔT preceding the convective onset decrease, suggesting that the "conduction" state is not of a purely diffusive nature, but rather involves some finite motion of the fluid. We believe that these are transient motions induced during part of the cycle, i.e., the unstable part, and subsequently damped out during the remainder of the cycle. This sort of behavior in modulated systems, and its effect on the measured heat transfer, are discussed in papers by Finucane and Kelly[4] and Davis.[3]

With regard to the absolute value of our measured critical Rayleigh numbers we refer to the theories of Hall[14] and Gershuni and Zhukhovitskii.[15] Both consider the case of an infinitely deep layer with one rigid boundary which is subject to temperature modulation. Hall has found that the stability equations relevant to a rapidly rotating cylinder in a uniform flow can be applied directly to the rapidly modulated convection problem, provided that the Prandtl number is unity. Calculations have produced a critical value of $N_{\mathrm{Ray}\,c} \simeq 108$ for that case, which is reasonably close to our measurement of $N_{\mathrm{Ray}\,c} = 122$ for Prandtl number $\sigma = 0.49$. Preliminary measurements at a higher Prandtl number, $\sigma = 0.75$, however, give $N_{\mathrm{Ray}\,c} \simeq 85$, so that our system appears to be less stable than that considered by Hall. On the other hand, considerable uncertainties in the He-I parameters, particularly in the viscosity, could lead to nearly 20% uncertainty in the absolute Rayleigh number. Effects due to the lateral boundaries can probably be neglected in this re-

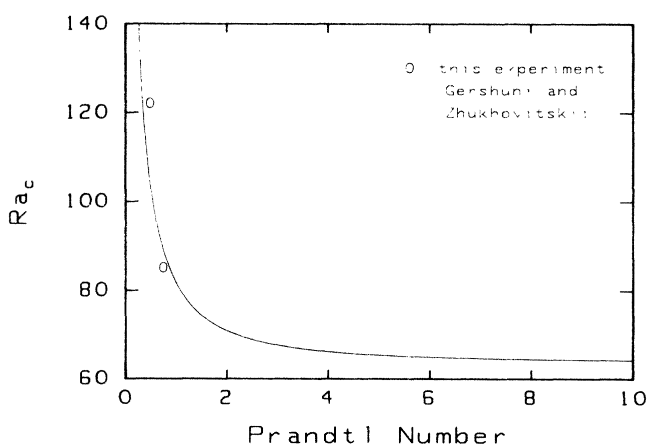

FIG. 4. $N_{\mathrm{Ray}\,c}$ as a function of the Prandtl number. Circles correspond to our experimental results for $\sigma = 0.49$ ($T = 2.63$ K) and $\sigma = 0.75$ ($T = 2.188$ K). The solid line represents analysis of theoretical calculations presented in Ref. 15.

gard, since the effective aspect ratio, defined as $\Gamma = D/2d_s$, where D is the diameter of our cell and d_s the boundary-layer thickness, is relatively large ($\Gamma \sim 20$ for $\sigma = 0.49$).

Apart from the absolute value, the trend of $N_{\mathrm{Ray}\,c}(\sigma)$ is clear and striking, varying nearly inversely with the Prandtl number. Our attention was drawn to the earlier theoretical study of Gershuni and Zhukhovitskii,[15] which is able to predict a lower bound for instability and the corresponding Prandtl-number dependence. This lower bound is found from minimization of an expression relating various parameters of the problem which holds at the stability boundary. Specifically, this expression is of the form $N_{\mathrm{Ray}}^2 = A + B/\sigma^2$, which implies that the critical Rayleigh number becomes nearly independent of Prandtl number when the latter is large, but that the flow becomes strongly stabilized as the Prandtl number is reduced below unity. Here the coefficients A and B are functions of the wave number of the convection, k, and another parameter a characterizing deep penetration of the disturbances. A double minimization with respect to a and k is performed (at fixed Prandtl number), and the results are shown as the solid line in Fig. 4, together with our experimental results. We refer the reader to Ref. 15 for specific details of the problem. Again, we wish to make clear that some caution must be exercised in making quantitative comparisons of the absolute Rayleigh numbers, considering the uncertainties involved. On the other hand, the trend of the Prandtl-number dependence, i.e., the strong stabilization accompanying a reduction in σ, is clear, and our measurements show gratifyingly close correspondence with the predicted behavior.

This experiment is capable of extension in several directions. Preliminary measurements show that the results are affected by rotation, and theoretical work by Hall is in progress to accompany the experiments. Also, the Prandtl-number dependence of $N_{\mathrm{Ray}\,c}$ can be extended to very low values of σ by use of mixtures of ^{3}He in superfluid ^{4}He, and to higher values of σ by means of classical fluids such as water (where visualization of the convection patterns would be possible).

After submitting this manuscript we received an interesting preprint containing further theoretical developments on the modulated Bénard problem.[16]

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