Inconsistency of Feynman Rules Derived via Path Integration

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We show that the Feynman rules in the covariant α gauge, the Coulomb gauge, and the axial gauge, in the generally adopted form derived via the Faddeev-Popov formalism, are inconsistent with one another. We suggest that they be replaced by the canonically derived ones, which are consistent. In particular, the Feynman rules in the Coulomb gauge contain the contribution of the anomalous Coulomb interaction, and the Feynman rules in the axial gauge with the principal-value prescription contain the contribution of ghost loops.

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In recent years, the formalism of path integration has become the standard way to quantize non-Abelian gauge field theories. This formalism enables one to obtain Feynmam rules for non-Abelian gauge field theories in various gauges, e.g., the covariant α gauge, the axial gauge, and the Coulomb gauge.[1] In this Letter, we shall show that such derived Feynman rules, in the forms generally adopted, are inconsistent with one another. We also show that canonical quantization leads to unambiguous Feynman rules in the Coulomb gauge as well as those in the axial gauge. These canonically derived rules are different from the corresponding ones derived via the straightforward application of the Faddeev-Popov formalism.[2] The former rules are also consistent with those in the covariant α gauge.

Let us begin with the Coulomb gauge. In 1962, Schwinger[3] derived, in the Coulomb gauge, the quantum Hamiltonian of non-Abelian gauge field theories.[4] This quantum Hamiltonian differs from the classical one as it contains additional terms originating from the ordering of operators. We shall call such additional terms those of anomalous Coulomb interaction. These anomalous terms, later rediscovered by Christ and Lee[5,6] in slightly different forms, are not included in the usual path-integral formulation.[1] We shall show that these anomalous terms are indeed needed for gauge invariance. Specifically, we have found that certain Feynman integrals in gauge-field theories have singular behaviors which render the conventional proof of gauge invariance invalid. This singular behavior cannot be handled by dimensional regularization, to name one method. By taking care of this singular behavior with cutoffs, we find the existence of additional terms which in the Coulomb gauge are precisely those given by the anomalous Coulomb interaction terms.

The gauge invariance of physical amplitudes can be established with the aid of the theorem of equivalence of Feynman rules[6,7]: Let the gluon propagator be

$$-i[g_{\mu\nu}-a_\mu(k)k_\nu-b_\nu(k)k_\mu]/(k^2+i\epsilon),$$

with $a_\mu(k)=-b_\mu(-k)$, and let the ghost-ghost-gluon vertex be $[(a\cdot k)-1]k^\mu-k^2a^\mu$, and the ghost propagator be $i/(k^2+i\epsilon)$; then the physical scattering amplitude is independent of $a_\mu(k)$. In particular, the set of Feynman rules with any $a_\mu(k)$ is equivalent to that with $a_u(k)=(1-\alpha)k_\mu/(2k^2)$, in which the gluon propagator is

$$-i[g_{\mu\nu}-(1-\alpha)k_\mu k_\nu/k^2]/(k^2+i\epsilon)$$

and ghost vertex is $-k^\mu$. This latter set is that in the covariant α gauge. There is, however, a point which appears to have been overlooked so far. The propagator for the gauge vector meson in the Coulomb gauge is given by

$$D^c_{\mu\nu}(k)=D^F_{\mu\nu}(k)+k_\mu\Delta_\nu+k_\nu\Delta_\mu, \tag{1}$$

where

$$\Delta_\nu(k)=\frac{i}{k^2+i\epsilon}\,\frac{k_0\delta_{\nu0}-\frac{1}{2}k_\nu}{\mathbf{k}^2}, \tag{2}$$

and where

$$D^F_{\mu\nu}(k)=-ig_{\mu\nu}/(k^2+i\epsilon) \tag{3}$$

is the propagator for the gauge vector meson in the Feynman gauge. In particular, for $\mu=\nu=0$, (1) gives $D^c_{00}(k)=i/|\mathbf{k}|^2$, which does not vanish in the limit $|k_0|\to\infty$ with $|\mathbf{k}|$ fixed. As a consequence, integrals over k_0 may be singular and must be handled with care. Indeed, if one ignored, incorrectly, this possibility of divergence, one would be led to conclude, with the aid of the theorem of equivalence, that the Feynman rules in the Coulomb gauge and those in the covariant α gauge, both derived via path integration, were equivalent. To take into account this singular behavior in k_0, we use, instead of D^c, the cutoff propagator

$$\bar{D}^c_{\mu\nu}(M,k)=D^F_{\mu\nu}(k)+(k_\mu\Delta_\nu+k_\nu\Delta_\mu)M^2/(M^2-k^2-i\epsilon), \tag{4}$$

where M is an introduced mass. This latter propagator vanishes in the limit $k_0 \to \infty$ with M and $\mathbf{k}$ fixed. It is therefore justified to conclude, with the aid of the theorem of equivalence of Feynman rules, that use of $\bar{D}^c$ together with the corresponding ghost-ghost-gluon vertex

$$\bar{C}_\mu \equiv \frac{i(q_\mu - q_0\delta_{\mu 0})}{\mathbf{q}^2}\frac{M^2}{M^2-q^2} + \frac{iq_\mu}{M^2-q^2} \qquad (5)$$

gives the same on-shell amplitudes as those given by the Feynman rules in the covariant α gauge. [The ghost propagator has been included in (5).] This is true for any M, and in particular for M very large. In the limit $M \to \infty$, $\bar{D}^c$ is equal to D^c. However, this does not mean that we may make the replacement of $\bar{D}^c$ by D^c (together with the replacement of ghost vertices) in the Feynman integrals. This is because the limit of an integral is not necessarily equal to the integral of the limit of its integrand. In other words, the limit $M \to \infty$ should be taken *after* the integration over k_0 has been carried out. It turns out that this makes no difference at the one-loop level. For example, we usually set the integral $\int_{-\infty}^{\infty} dq_0\, q_0/(q^2+i\epsilon)$ equal to zero by the reason of symmetric integration. This conclusion does not change if we introduce a cutoff factor of $M^2/[M^2-(q+k)^2]$, say. This can be easily proved if we express the integral over q_0 by one over time:

$$\int_{-\infty}^{\infty} \frac{dq_0}{2\pi}\frac{q_0}{q^2}\frac{M^2}{M^2-(k+q)^2} = \int_{-\infty}^{\infty} dt\, F(t)\delta_M(t)e^{-ik_0t}, \qquad (6)$$

where $F(t)$ and $\delta_M(t)$ are the Fourier transforms (with respect to time) of q_0/q^2 and $M^2/(M^2-q^2+i\epsilon)$, respectively. More precisely,

$$F(t) = -\tfrac{1}{2}ie^{-i|\mathbf{q}||t|}\epsilon(t) \qquad (7)$$

and

$$\delta_M(t) \simeq \tfrac{1}{2}iMe^{-iM|t|}, \quad M >> 1, \qquad (8)$$

where

$$\epsilon(t) = t/|t|. \qquad (9)$$

Because of the factor $e^{-iM|t|}$ in (8), the dominant integration region for the time integral in (6) is $|t| = O(1/M)$. Thus we may, in the limit $M \to \infty$, approximate e^{-ik_0t} in (6) by unity. Since $F(t)$ is an odd function of t and $\delta_M(t)$ is an even function of t, the integral on the right-hand side in (6) approaches zero in the limit $M \to \infty$.

This is, however, no longer true for diagrams of two or more loops. Consider, for example, the double integral

$$\int \frac{k_0q_0}{(k^2+i\epsilon)(q^2+i\epsilon)(\mathbf{k}+\mathbf{q})^2}dq_0\,dk_0, \qquad (10)$$

which appears in the scattering amplitude corresponding to the diagram in Fig. 1(a). Again, this integral is usually set equal to zero by symmetric integration. However, this is no longer true once we introduce the cutoff factor $M^2/[M^2-(k+q)^2]$. This is because each of k_0/k^2 and q_0/q^2 contributes a factor $\epsilon(t)$. Since $\epsilon^2(t)=1$ is an even function of t, the corresponding integral over t is nonzero in the limit $M \to \infty$. Therefore, the anomalous terms first appear at the two-loop level. Indeed, we have explicitly confirmed that the g^2 term of the vacuum diagrams and the g^4 term of the gluon self-energy diagrams contain precisely the contribution of the anomalous terms of Schwinger.

Next we turn to the axial gauge. In the temporal axial gauge, the gluon propagator in the momentum space is singular at $k_0=0$. (There is a similar singularity in the case of spatial axial gauge.) The path-integral formulation is unable to determine precisely how to define the axial-gauge propagator in the neighborhood of $k_0=0$. One prescription is to use the principal value. Such a prescription has always been suspect, and for a good reason—it is wrong. To see this, let us note that we may use, by the theorem of equivalence of Feynman rules, the following gluon

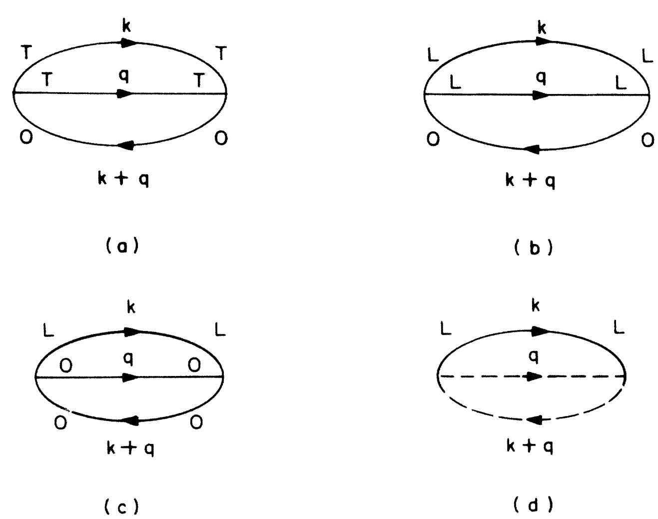

FIG. 1. Solid lines represent gluons, with T, L, and 0 denoting transverse, longitudinal, and temporal polarizations, respectively. Dashed lines represent ghosts.

propagator:

$$\tilde{D}^{ab}_{\mu\nu} = -\frac{i\delta_{ab}}{k^2+i\epsilon'}\left[g_{\mu\nu} - (\delta_{\mu 0}k_\nu + \delta_{\nu 0}k_\mu)\frac{1}{2}\left(\frac{1}{k_0+i\epsilon}+\frac{1}{k_0-i\epsilon}\right) + k_\mu k_\nu \frac{1}{2}\left(\frac{1}{(k_0+i\epsilon)^2}+\frac{1}{(k_0-i\epsilon)^2}\right)\right], \quad (11)$$

together with the ghost-ghost-gluon vertex

$$\frac{-ik^2k_0n_\mu + \epsilon^2 k_\mu}{(k^2+i\epsilon')(k_0^2+\epsilon^2)}. \quad (12)$$

[The ghost propagator has been included in (12).] In the limit $\epsilon \to 0$, this propagator is traditionally regarded as the temporal axial-gauge propagator in the principal-value prescription. Furthermore, the expression in (12) becomes, in the limit $\epsilon \to 0$,

$$-in_\mu \mathrm{P}(1/k_0).$$

Since

$$\int_{-\infty}^{\infty} dk_0 \,\mathrm{P} \prod_{n=1}^{N} \frac{1}{k_0+a_n} = 0,$$

for $N \geq 2$ and $a_n \neq a_m$ if $n \neq m$, it may appear that one need not, in the axial gauge, take into account ghost loops, as asserted by the path-integral approach. This conclusion is again erroneous. This is because the Feynman integrations are singular at $k_0 = 0$ if ϵ is set equal to zero; thus the limit $\epsilon \to 0$ must again be handled with care. It is interesting to compare the situation here with that in the Coulomb gauge, where the integration is singular at $k_0 = \infty$. In that case, we introduced the cutoff parameter M, setting $M \to \infty$ only after integration. In the axial gauge, the integration is singular at $k_0 = 0$, and the parameter ϵ in (11) and (12) serves the same function as M in the Coulomb gauge. Therefore we must take the limit $\epsilon \to 0$ after the integration has been carried out, not before. In other words, the limit $\epsilon \to 0$ should be taken for the Feynman *integral*, not the *integrand*. As a result, both the ghost and the gluon with temporal polarization give nonzero contributions to the scattering amplitude. Indeed, the contributions of the ghost and the gluon of temporal polarization also appear at the two-loop level. Some examples are illustrated in Figs. 1(b)–1(d). It is easy to see that the amplitudes corresponding to these diagrams, not included in the traditional principal-value prescription, are nonzero. To verify this for diagram 1(b), we note that in the integration region where k_0 and q_0 are both of the order of ϵ, $\tilde{D}_{LL}$ and $\tilde{D}_{00}$ of (11) are of the order of ϵ^{-2} and ϵ^0, respectively. Furthermore, a vertex factor in Fig. 1(b) is of the order of ϵ. Therefore, the contribution of this integration region is of the order of unity as $\epsilon \to 0$. For diagram 1(c), there is one fewer $\tilde{D}_{LL}$, but this is compensated for by each of the vertex factors being of the order of unity instead of ϵ. In diagram 1(d), we need to keep only the second term in the numerator of (12) for the ghost-ghost-gluon vertex, and the rest of the consideration is the same as that for diagram 1(c).

It is possible to choose the axial-gauge propagator to be[8]

$$\tilde{D}^{ab}_{\mu\nu} = -\frac{i\delta_{ab}}{k^2+i\epsilon'}\left[g_{\mu\nu} - \frac{n_\mu k_\nu}{k_0+i\epsilon} + \frac{n_\nu k_\mu}{k_0-i\epsilon} + \frac{k_\mu k_\nu}{k_0^2+i\epsilon}\right]. \quad (13)$$

The ghost-ghost-gluon vertex corresponding to (13) is

$$(-k^2n^\mu - i\epsilon k_\mu)/(k_0+i\epsilon). \quad (14)$$

For such a vertex, the integrand has no singularity at $k_0 = i\epsilon$. Therefore, we may choose to close the contour integration in the upper-half k_0 plane. In other words, there is, in this prescription, no contribution from ghosts. However, we still have to take into account the contribution of temporal gluons.

Finally, we mention that it is possible to choose the gluon propagator in such a way that $D_{0\mu}$ is strictly zero for all μ. From the correspondence formula between a scattering amplitude in the temporal axial gauge and that in the Feynman gauge,[9] we find that an alternative prescription is as follows: For the propagator of the longitudinal gluon,

$$D_{LL}(x,x') = -i\delta^{(3)}(\mathbf{x}-\mathbf{x}')t_>, \quad (15)$$

where $t_>$ is the larger of x_0 and x_0'; for the transverse gluon,

$$D^T_{ij}(x,x') = i\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2+i\epsilon'}\left(\delta_{ij} - \frac{k_ik_j}{\mathbf{k}^2}\right). \quad (16)$$

In addition

$$D_{0\mu}(x,x') = D_{\mu 0}(x,x') = 0, \quad (17)$$

and there is no contribution from ghost loops. Note that the propagator (15) is not invariant under time translation, and therefore has no Fourier transform. It is also the propagator given by Caracciolo, Curi, and Menotti[10] in their study of the Wilson loop.

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[1]See, for example, E. S. Abers and B. W. Lee, Phys. Rep. **9C**, 5 (1973); W. Marciano and H. Pagels, Phys. Rep. **36C**, 139 (1978).

[2]L. S. Faddeev and U. N. Popov, Phys. Lett. **25B**, 29 (1967).

[3]J. Schwinger, Phys. Rev. **127**, 324 (1962), and **130**, 402 (1963).

[4]C. N. Yang and R. L. Mills, Phys. Rev. **96**, 191 (1954).

[5]N. H. Christ and T. D. Lee, Phys. Rev. D. **22**, 939 (1980).

[6]H. Cheng and E. C. Tsai, "Canonical Quantization of Non-Abelian Gauge Field Theories and Feynman Rules" (to be published).

[7]H. Cheng and E. C. Tsai, "The Anomalous Coulomb Interaction" (to be published).

[8]H. Cheng and E. C. Tsai, "Ghostless Feynman Rules in Non-Abelian Gauge Theories" (to be published).

[9]See Eq. (6) in H. Cheng and E. C. Tsai, "Correspondence between Quantum Gauge Theories without Ghost Fields and Their Covariantly Quantized Theories with Ghost Fields" (to be published). In this equation, we may replace the factor $\exp[-\int(\nabla\cdot\mathbf{A}^a)\nabla^{-2}(\nabla\cdot\mathbf{A}^a)d^3x]$ by $\delta(\nabla\cdot\mathbf{A}^a)$. See the discussion preceding Eq. (13) in this reference. To derive the Feynman rules, we have to use the adiabatic hypothesis and turn off the coupling constant at $|t|=\infty$. Thus the determinant in Eq. (6) in this reference can be set equal to unity.

[10]S. Caracciolo, G. Curi, and P. Menotti, Phys. Lett. **113B**, 311 (1982).