0^{++}-2^{++} Glueball Mass Ratio in Non-Abelian Gauge Theories

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By means of a high-statistics Monte Carlo simulation, Fisher scaling is studied for pure SU(2) and SU(3) lattice gauge theory. In both cases 2^{++} is found to be the lowest state above the vacuum, and the $m(0^{++})/m(2^{++})$ glueball mass ratio is slightly larger than 1. For SU(2), Monte Carlo data smoothly continue the analytic $m(0^{++})/m(2^{++})$ small-volume calculation towards the infinite-volume continuum limit. New results for the ratio between the square root of the string tension and the mass gap are also reported.

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Glueballs[1] are the particles of the pure gauge sector of quantum chromodynamics (QCD). Theoretically as well as experimentally they have remained a challenging problem. Lattice gauge theories[2] provide us with a regularization scheme that allows (in principle) non-perturbative calculations. For SU(N) lattice gauge theory (without quarks) one has to calculate averages of gauge-invariant operators with respect to the partition function

$$Z=\int \prod_l dU(l)\exp\{-N^{-1}\beta\sum_p \mathrm{Re\,Tr}[1-U(\dot{p})]\}. \tag{1}$$

The product is over all links l of a four-dimensional hypercubic lattice, and $U(l)\in \mathrm{SU}(N)$. For each plaquette p, $U(\dot{p})$ is the ordered product of the four link matrices surrounding the plaquette and dU is the SU(N) Hurwitz measure. In this Letter we report Monte Carlo (MC) results from simulations on L^3L_t, $L_t \gg L$, lattices with periodic boundary conditions.

We are interested in the continuum limit, where the lattice regularization becomes removed. The continuum limit is obtained for $\beta\to\infty$, $L\gg\xi$, where ξ is the correlation length. The physical mass scale is set by the mass gap m, defined to be the smallest energy eigenstate above the vacuum. For β large the lattice spacing a goes to zero like

$$a\sim\beta^{51/121}m^{-1}\exp\left(-\frac{12\pi^2}{11N^2}\beta\right), \tag{2}$$

and mass ratios approach constant values according to

$$\frac{m_1}{m_2}=\mathrm{const}_{12}+O\left[\frac{1}{\xi^2}\ln\left(\frac{1}{\xi}\right)\right]. \tag{3}$$

The lowest mass above the vacuum is called mass gap m, and its inverse defines the correlation length $\xi=(am)^{-1}$. We shall find $m=m(2^{++})$.

Strong-coupling expansions[3] have been carried out in the Hamiltonian as well as in the Euclidean formulation of lattice gauge theories. These calculations are valid for small correlation length. Then extrapolations to the physical limit $\xi\to\infty$ are done. In these calculations $m(0^{++})$ comes out to be the mass gap. The $m(2^{++})$ mass is only slightly higher and typically one finds $m(0^{++})/m(2^{++})\approx 0.9$. Strong-coupling expansions, however, diverge already at a very small correlation length $\xi<1$ and their continuum extrapolations are therefore questionable.

Previous MC calculations[4,5] attempted to follow the mass spectrum beyond the validity of the strong-coupling expansion towards larger correlation length. Masses were estimated from correlations between various Wilson-loop operators. For the 0^{++} state these calculations were partially successful. The crossover from the strong-coupling behavior to the weak-coupling behavior (2) is clearly indicated around $\xi\approx 0.6$, and rather reliable results could be obtained up to $\xi\approx 1.2$. On the other hand, estimates for masses of excited states seemed not to follow this crossover from strong to weak coupling behavior. Instead, these mass estimates suffered from large statistical noise at distances as small as already $t=1$, where t is the Euclidean-time separation between appropriate operators. This suggested the interpretation that excited glueball states develop large masses in units of $m(0^{++})$. But the status was never satisfactory, as an equivalent interpretation is to attribute the results to bad trial wave functions for the excited states. In particular, the disappearance of the correlations into statistical noise at short distances made consistency checks (from correlations at larger distances) impossible. Even worse, enhancing the signal by means of a specialized 2^{++} source[6] exhibited strong instabilities at distance $t=3$ without allowing conclusions about the asymptotic ($t\to\infty$) mass value.

We here report results from reanalysis of the problem along an independent line of reasoning. We systematically study the approach to the continuum limit in Fisher's[7] scaling variable

$$z=L/\xi. \tag{4}$$

L is the edge size of the spacelike volume, and the extension in time direction, L_t, is taken to be infinite. For L sufficiently large, mass ratios will approach a universal (L-independent) curve $r(z)$. Along this curve the infinite-volume continuum limit is approached for $z \rightarrow \infty$, whereas small z values allow analytic small-volume calculations. With the SU(2) gauge group, spectrum calculations for small z were done by Lüscher and Münster,[8] who find their expansion to break down at $z \approx 1.5$. Monte Carlo calculations for small and moderate z became feasible due to the finding[9] that correlation functions between spatial Polyakov loops in the adjoint SU(2) representation give an excellent signal for glueball masses. Polyakov[10] loops are Wilson loops along straight lines which are closed by periodic boundary conditions; for instance, $P^x(y,z,t)$ is a loop in the x direction. They allow Dobrushin-Lanford-Ruelle–improved[11] measurements which, in turn, are partially responsible for the good signal.

On a hypercubic lattice, different spin states may be distinguished by our projecting out appropriate irreducible representations of the cubic group.[3,4] But, as a result of limited time for software development, the MC calculations of Ref. 9 were done for the correlation function of the zero-momentum operator

$$O^x(t) = \sum\nolimits_{yz} P^x(y,z,t). \tag{5}$$

This operator couples to 0^{++} as well as 2^{++}. In previous MC investigations[4,5] strong signals were exclusively obtained for 0^{++}. In reliance on this experience, the results[9] were misinterpreted to be mainly 0^{++}. For our present investigation we have built appropriate linear combinations of adjoint Polyakov loops, projecting on the 0^{++} and the 2^{++} channels:

$$O^{0^{++}}(t) = O^x(t) + O^y(t) + O^z(t), \tag{6a}$$

$$O^{2^{++}}(t) = O^x(t) - O^y(t). \tag{6b}$$

We further extended our calculations to the SU(3) gauge group. The high statistics of our numerical results is summarized in Table I. A sweep is defined by the updating of each link on the lattice once. As in Ref. 9, the SU(2) gauge group is approximated by the icosahedral subgroup and a Metropolis program with six hits per update is used. In case of the SU(3) gauge group we run a Metropolis program with ten hits per update. The first 2000–5000 sweeps are used for reaching equilibrium and are discarded with respect to measurements. For practical reasons the approximation of $L_t = \infty$ varies between $L_t = 24$ and $L_t = 64$. After each few sweeps normal measurements for the considered correlations functions are done, whereas Dobrushin-Lanford-Ruelle–improved[11] measurements are done approximately every twenty sweeps. Details will be published elsewhere. They vary for different β values and lattices.

For a single mass, the crossover from the small-volume to the asymptotic ($z \rightarrow \infty$) behavior is extremly rapid and, therefore,[9] prohibitive against quantitative matching of analytic results with MC data. As we will demonstrate, this statement does not hold for mass ratios. Figure 1 summarizes our SU(2) MC results, and corresponding SU(3) results are given in Fig. 2. The procedure used to extract masses was described in Ref. 9. In Fig. 1 our $m(0^{++})/m(2^{++})$ MC data smoothly continue the analytic calculation, which already gives an approximately straight line with

$$r(z) = m(0^{++})/m(2^{++}) \approx 1.2, \tag{7}$$

to larger z values. Different lattice sizes and β values merge (within statistical errors) to one curve $r(z)$, hence supporting the proposition that Fisher scaling already holds with our rather small-sized lattices. At large z values our data seem to indicate an approach towards $m(0^{++})/m(2^{++}) \approx 1.0$, but one has to be careful with the interpretation. Presently, our large z values mainly rely on rather small β values, because we can easily reach large z by decreasing the correlation length

$$\xi = [am(2^{++})]^{-1} \tag{8}$$

in Eq. (2). At small β values our data then become contaminated by the nonuniversal strong-coupling domain, where $m(0^{++})/m(2^{++}) = 1$ holds for $\beta \rightarrow 0$.

TABLE I. Monte Carlo statistics.

	SU(2)			SU(3)	
L^3L_t	β	10^3 sweeps	L^3L_t	β	10^3 sweeps
4^324	2.25	752	4^332	5.8	69
4^324	2.40	244	4^332	6.0	53
4^364	2.70	182	4^332	6.2	107
			4^332	6.4	107
6^332	2.40	294	4^332	6.6	107
8^332	2.70	168	6^332	5.8	95
8^332	3.00	96	6^332	6.0	95
			6^332	6.2	95

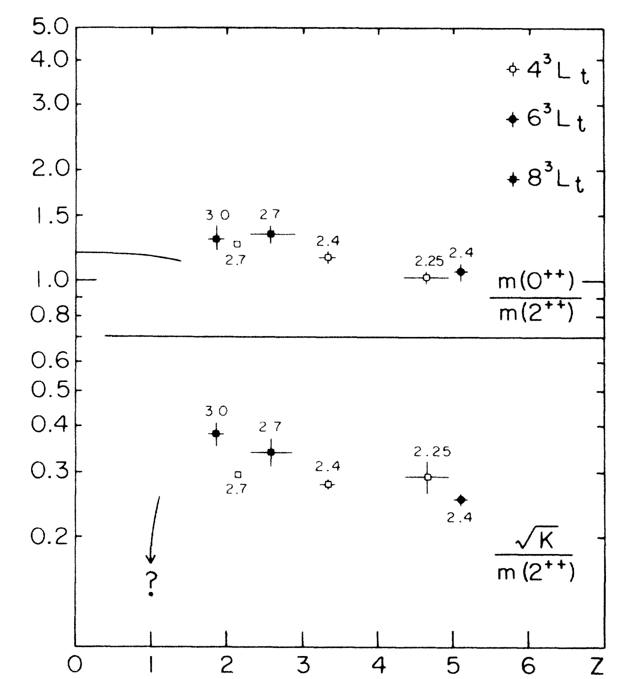

FIG. 1. SU(2) data for $m(0^{++})/(2^{++})$ mass ratios (upper part) and for $\sqrt{K}/m(2^{++})$ (lower part). The straight line in the upper part is the small-volume expansion (Ref. 8) in the range where it may be trusted. The arrow in the lower part denotes the z value, at which tunneling contributions are claimed (Ref. 12) to become important. β values are explicitely given.

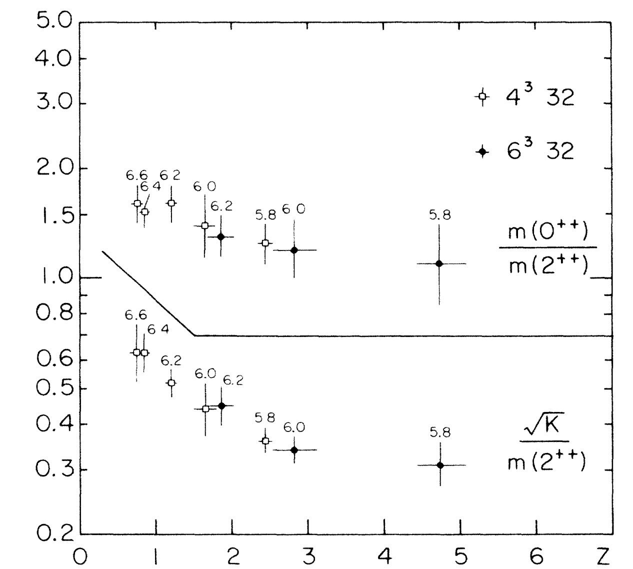

FIG. 2. SU(3) data for $m(0^{++})/m(2^{++})$ mass ratios (upper part) and for $\sqrt{K}/m(2^{++})$ (lower part). β values are explicitely given.

Nonuniversal effects will disappear when one goes at constant z to larger L and β values. Ideally, we will be able to study these finite-size scaling corrections numerically, but we have not yet succeeded in pushing our calculations far enough. In conclusion, an $m(0^{++})/m(2^{++})$ value slightly larger than 1 is most supported. Where possible, we have checked on our calculation of Ref. 9 and we found that the previously estimated masses are compatible with present $m(2^{++})$ estimates. The admixture of 0^{++} accounts for a small bias of always less than 4%. Qualitatively, our SU(3) $m(0^{++})/m(2^{++})$ results of Fig. 2 are similar to those for SU(2). Towards small z values there is a tendency of $m(0^{++})/m(2^{++})$ to increase. Unfortunately, the analytic small-volume expansion does not exist (work on it is in progress[13]). Our preliminary results[14] were for SU(3) only.

In the lower parts of Figs. 1 and 2 we present new string tension results.[15] Measurement of the correlation functions of Polyakov loops in the fundamental representation allows determination of the energy E of a 't Hooft[16] electric flux of length L. The 't Hooft string tension is defined as $K=E/L$ and was numerically first studied in Ref. 11. We plot results for

$$\sqrt{K}/m(2^{++}). \tag{9}$$

They are approximately proportional to the $m(0^{++})/m(2^{++})$ ratios, implying that $\sqrt{K}/m(0^{++})$ is nearly constant. For large z an approach to $\sqrt{K}/m(2^{++})\approx 0.3$ [slightly smaller for SU(2)] is indicated (without our taking into account the possible Coulomb correction). Our results can be translated into megaelectronvolts by the assumption $\sqrt{K}\approx 420$ MeV.

Very recently, Koller and van Baal[12] did a small-volume SU(2) calculation for tunneling contributions from the effective Hamiltonian of Ref. 8. They predict $\sqrt{K}/m(0^{++})$ to drop down to approximately zero for z less than ≈ 1.2. If this can be confirmed by MC data, a better understanding of the crossover from small z to large z is achieved. Small z and reasonably large L need high β values. As a result of using the icosahedral subgroup, we were at the moment unable to check their SU(2) prediction. For SU(3) our smallest z values are around $z\approx 0.7$ and we have no signal for $\sqrt{K}/m(0^{++})$ falling down to zero. Another limitation, showing up at high β values, is metastabilities and the fact that the projection of adjoint Polyakov loops on glueball wave functions becomes bad. Consequently, even for the full group, we may encounter problems when probing for very small z values. Tunneling contributions for the $m(0^{++})/m(2^{++})$ ratio should be calculated. They might explain the small jump seen in Fig. 1 between the analytic $m(0^{++})/$

$m(2^{++})$ results and our small-z MC data.

In conclusion, MC data for correlation functions in non-Abelian gauge theories have been improved by several orders of magnitude. Consequently, spectrum calculations are now much more reliable. The calculations we propose include consistent use of Fisher's scaling variable (3) which ensures that we are approaching the infinite-volume continuum limit $z \to \infty$ on a universal curve. As results from different lattices and β values have to fall on one curve, a consistency check for finite-size limitations is included. The $m(0^{++})/m(2^{++})$ mass ratio MC data for the first time smoothly continue analytic results, namely the small-volume expansion of Ref. 8. This gives more confidence in the MC results as well as in the hope that the asymptotic behavior for $z \to \infty$ is indeed reached.

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