Third Law of Black-Hole Dynamics: A Formulation and Proof

W. Israel[(a)]
Research Institute for Fundamental Physics, Yukawa Hall, Kyoto University, Kyoto 606, Japan
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It is shown that no continuous process in which the energy tensor of accreted matter remains bounded and satisfies the weak energy condition in a neighborhood of the apparent horizon can reduce the surface gravity of a black hole to zero within a finite advanced time. This gives a precise expression to the third law of black-hole mechanics.

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It was only in a quantum setting, with the discovery of the Hawking radiation in 1974,[1] that the real meaning of a black-hole "temperature" emerged. However, it had been noted considerably earlier[2] that the classical laws of black-hole dynamics (BHD) were curiously similar to those of thermodynamics, with the area and surface gravity κ of the horizon playing the roles of entropy and temperature, respectively. Bardeen, Carter, and Hawking (BCH),[3] in the most complete expression of this parallelism worked out to date, offer the third law only as an unproven and not very precisely worded conjecture, and there is a prevailing impression that the relatively problematical status of this law in thermodynamics necessarily extends to BHD. It is the object of this paper to dispel this idea.

In thermodynamics, the third law has been formulated in a variety of ways.[4] Two (essentially equivalent) formulations, due to Nernst, state that (1) isothermal reversible processes become isentropic in the limit of zero temperature, and (2) the temperature of a system cannot be reduced to zero in a finite number of operations. A stronger version, due to Planck, states the following: The entropy of any system tends, as $T \to 0$, to an absolute constant, which may be taken as zero.

In BHD there is no analog of Planck's version. The formulation proposed by BCH is patterned after Nernst's unattainability principle: "It is impossible by any process, no matter how idealized, to reduce κ to zero in a finite sequence of operations."

The status of this principle in the two theories has to be assessed in the light of an essential difference between them. The laws of thermodynamics, unlike those of BHD, are independent of other macroscopic physical laws, and cannot be deduced from them, or from each other. (For example, the second law requires only that reversible processes at zero temperature be adiabatic, not necessarily isentropic.) Justification for the thermodynamical third law can therefore come only from empirical evidence or from statistical mechanical considerations, the latter requiring a microphysical model for each system and prone to uncertainties involving the enumeration of degenerate ground states. Black-hole states, in contrast, are completely determined by macroscopic variables (their surface geometry), whose evolution is governed by the Einstein field equations. Just as the second law of BHD finds rigorous expression in Hawking's area theorem,[2] there is no reason why the third law should not admit a clearcut *dynamical* formulation and proof. BCH stopped short of this step, perhaps because their analysis was largely confined to quasistatic processes, in terms of which it is difficult to spell out exactly what is meant by "finite sequence of operations." In a dynamical context, it becomes possible to define a "process" or "operation" as an interaction between a black hole and its environment whose active phase occupies a finite interval of advanced time (i.e., a finite time in the experience of free-falling observers near the horizon.)

Dynamical studies of spherical models[5-7] show that the third law can be violated if the black hole is allowed to absorb material which does not have positive energy density and a reasonable degree of smoothness while crossing the *apparent* horizon. Infractions can result from the absorption of infinitesimally thin, massive shells,[5] which force the apparent horizon to jump outward discontinuously; in fact, an extremal ($\kappa = 0$) Reissner-Nordström black hole can be created outright at a finite advanced time by implosion of an extremally charged ($|Q| = M$) hollow spherical shell.[6] Injection of matter whose energy density is or becomes negative in a neighborhood of the apparent horizon can violate not only the third law, but cosmic censorship as well.[7] These counterexamples serve as useful guiding constraints on attempts to give a precise form to the third law.

Insight into the mechanism of extremization can be gained from Fig. 1, which portrays two ways of arriving at an intermediate stage of this process for a charged spherical black hole. In both scenarios, absorption of a (thick) shell of charged material causes the charge-to-mass ratio $|Q|/M$ of the hole to rise continuously to a higher (but still nonextremal) value. [The diagrams should also give a qualitative picture of more general (nonspherical) situations, including the

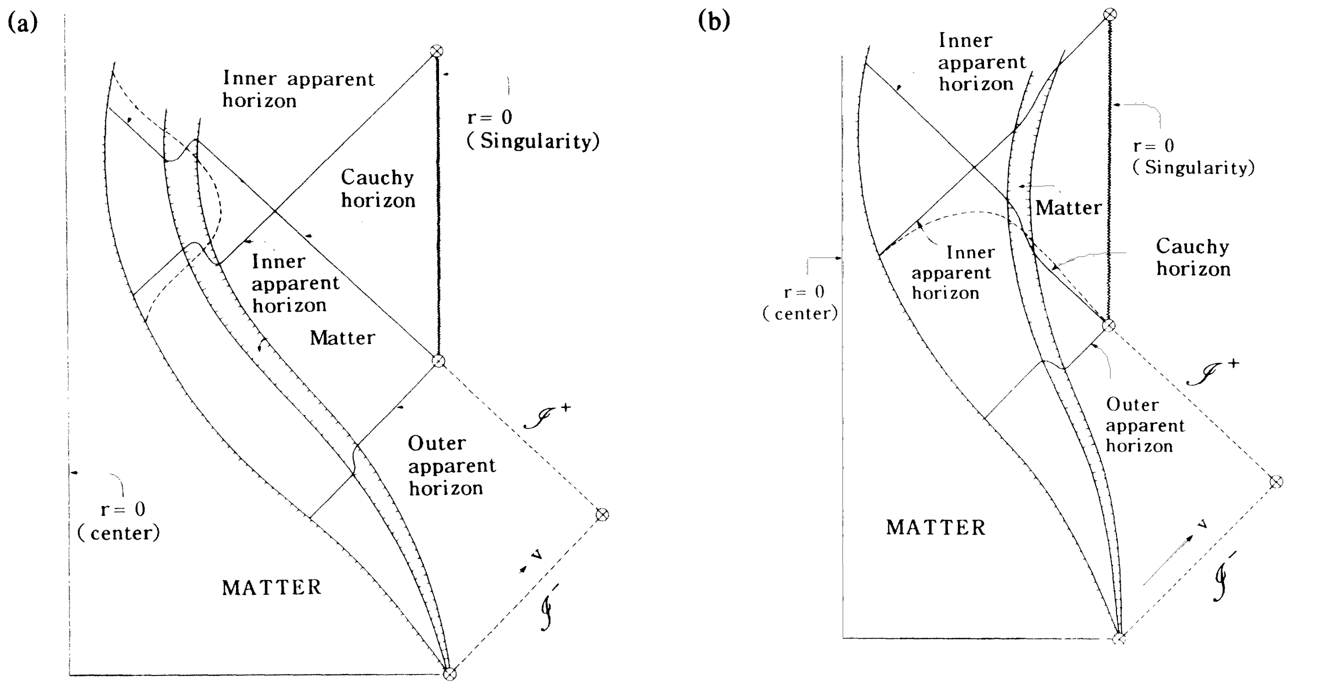

FIG. 1. Two scenarios for raising the charge-to-mass ratio of a charged spherical black hole by injection of a spherical shell of charged material, which may fall into either of the two sectors of the region within the inner horizon. The black hole is assumed to have been originally formed by collapse of the charged matter on the extreme left. In (a) the injected shell must have negative energy density, at least while crossing the inner apparent horizon. This can be seen by noting that an outgoing pencil of radial light rays that crosses the outgoing sheet of the inner apparent horizon changes there from converging to diverging, whereas a positive energy density would accelerate the convergence (Raychaudhuri's equation, Ref. 11). The dashed curve in each part represents a three-cylinder Σ, the rigid ($r = \text{const}$) extension of a trapped surface S_0 situated near the lower left-hand side of the curve. Both cylinders change their intrinsic signatures from spacelike to timelike upon crossing the inner apparent horizon, in apparent contravention of the lemma. What breaks down is the energy condition (iii) of the lemma in the case of (a), whereas in (b) it is the ingoing convergence condition (i).

injection of angular momentum into spinning black holes, provided they are cut off at the Cauchy horizon, whose compact sector is unstable to time-dependent perturbations.[8]] The effect of this absorption is to inflate the inner horizon. If the process is continued until the inner and outer horizons merge, thus squeezing out all trapped surfaces, extremization will have been achieved. It is evident from the figures that this can only happen at a finite advanced time if the injected material falls through the *outgoing* sheet of the inner horizon [Fig. 1(a)] before reaching the Cauchy horizon. However, as explained in the caption, this requires the injected material to have negative energy density, at least while crossing the inner apparent horizon. In the alternative scenario [Fig. 1(b)], involving influx of positive energy through the *ingoing* sheet of the inner horizon, extremization is deferred until the outer horizon meets this sheet, necessarily at infinite advanced time.

These simple considerations suggest a general formulation of the third law, which will first be stated informally.

A nonextremal black hole cannot become extremal (i.e., lose its trapped surfaces) at a finite advanced time in any continuous process in which the stress-energy tensor of accreted matter stays bounded and satisfies the weak energy condition in a neighborhood of the outer apparent horizon.

This can be restated somewhat more formally, though inevitably the mathematical perfection of Hawking's characterization of the second law[2] cannot be emulated, since the apparent horizon does not share the event horizon's clean causal properties. To facilitate such a formulation it is convenient to define a black-hole space-time as being strongly future asymptotically predictable in an "extended" sense if the closure (in the conformally completed manifold) of the domain of dependence $D^+(\mathcal{S})$ of a partial Cauchy surface $\mathcal{S}$ contains not only $\mathcal{I}^+$ and a complete future segment of the event horizon (the usual definition[9]), but also the outermost trapped surfaces. (For a black hole perturbed by smooth processes that taper off at late times, the event and outer apparent horizons will never be widely separated and will approach each other asymptotically, so that the extension involved is hardly significant.) It is then possible[9] to foliate $D^+(\mathscr{S})$ by a nest of partial Cauchy surfaces $\mathscr{S}(\tau)$ intersecting $\mathscr{J}^+$, such that, for each positive τ, $\mathscr{S}(\tau)$ is a complete Cauchy surface for the portion of $D^+(\mathscr{S})$ lying to its future. The (nonunique) time function τ thus defined globally over $D^+(\mathscr{S})$ can be viewed as an "advanced time" for a network of observers ringing the horizon.

The third law may now be phrased as follows. In a strongly future asymptotically predictable black-hole space-time, let there be a continuous process [i.e., $g_{\alpha\beta} = (C^1$, piecewise $C^3)$ in Lichnerowicz admissible coordinates] in which $\mathscr{S}(\tau)$ contains trapped surfaces for all $\tau < \tau_1$, but none for $\tau > \tau_1$. Then the weak energy condition is necessarily violated in a neighborhood of the apparent horizon on $\mathscr{S}(\tau_1)$.

A simple way to establish this is to apply the following lemma,[10] which in essence is just a convenient restatement of Raychaudhuri's equation.[11]

Lemma.—Let S_0 be a trapped two-surface, and consider an extension of S_0 to a three-cylinder Σ, foliated by two-sections $S(\tau)$, which has the following properties: (i) The extension is "semirigid," which means that it is locally area preserving [elements of two-area are preserved under Lie transport along the normal to $S(\tau)$], and such that *ingoing* light beams orthogonal to $S(\tau)$ converge; (ii) Σ is regular; (iii) the weak energy condition holds on Σ. Then Σ is everywhere spacelike, so that all subsequent two-sections $S(\tau)$ are trapped.

This lemma was previously invoked in an attempt to establish a property, called gravitational confinement,[8] that would permanently restrain collapsing material enclosed within a trapped surface (which must include a singularity) from causally influencing the environment. In the case of gravitational collapse, the regularity condition (ii) constitutes a serious drawback of the approach, since one is forced to postulate a not-inconsiderable part of what one would really like to prove. For application to the third law, however, this is no longer of consequence, since, as previously noted, a regularity assumption is a *necessary* part of the input.

The third law is now an immediate consequence of the lemma if we choose for S_0 one of the outermost trapped surfaces on $\mathscr{S}(\tau_1 - \epsilon)$. Extending S_0 semirigidly to the future and assuming (iii) to hold leads at once to the desired contradiction.

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[a]On leave of absence from Theoretical Physics Institute, Avadh Bhatia Physics Laboratory, University of Alberta, Edmonton, Alberta T6G 2J1, Canada.

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