Dynamics of the ^{3}He A-B Phase Boundary

S. Yip and A. J. Leggett
Department of Physics, University of Illinois at Urbana-Champaign, Urbana, Illinois 61801
(Received 10 February 1986)

We investigate the friction coefficient of the moving boundary. At not too low temperatures the friction is ascribed primarily to transmission and Andreev reflection of quasiparticles; the resulting theoretical value of the terminal velocity v_{AB} is in reasonable agreement with the data of Buchanan, Swift, and Wheatley. At lower temperatures we predict a saturation of v_{AB} and underdamped oscillations of the pinned boundary.

PACS numbers: 67.50.Fi, 64.70.Ja

The phase transition between the A and B phases[1,2] of liquid ^{3}He is first order, and its equilibrium temperature T_{AB} is a strong function of pressure P and magnetic field H. Since, moreover, the A phase supercools very appreciably, it should be possible to study the structure and behavior of the A-B phase boundary under a wide variety of conditions. Indeed, its static properties have been investigated both experimentally[3] and theoretically.[3-6]

In this Letter we outline the general physical principles which we believe govern the *dynamics*[7] of the phase boundary, and apply them to discuss quantitatively its terminal velocity in the regime investigated by Buchanan, Swift, and Wheatley.[8] We also discuss more briefly and qualitatively the behavior at low temperatures. We hope to supply more details[9] elsewhere.

Essential to our argument are the following order-of-magnitude inequalities, which hold over almost all of the region of interest[10]: (1) The width of the boundary, $d(T)$, which is[4,5] a few times the temperature-dependent coherence length[2] $\xi(T)$, is much shorter than the mean free path $l(T)$ of an excitation in either bulk phase (cf. Greaves and Leggett[11]). (2) The velocity of the boundary, v, is small compared to the Fermi velocity v_F (cf. Ref. 8, and below). (3) The characteristic time associated with the motion of the boundary, which is of order d/v for the experiment of Ref. 8 and of order of the reciprocal vibrational frequency for the pinned boundary, may be either the reciprocal vibrational frequency for the pinned boundary, may be either large or small compared to the quasiparticle collision time $\tau(T) \sim l(T)/v_F$, but is always long compared to the characteristic adjustment time $\hbar/\Delta(T)$ of the Cooper pairs, where $\Delta(T)$ is the rms gap in either bulk phase.

In general, if we consider the displacement $\mathbf{u}$ of a small area of the boundary under conditions (2) and (3), we should expect it to satisfy an equation of the general form

$$M^*\ddot{\mathbf{u}} + \Gamma\dot{\mathbf{u}} = \mathbf{F} \qquad (1)$$

where M^*, Γ, and $\mathbf{F}$ are respectively the inertial mass, friction coefficient, and external (conservative) force per unit area of the boundary. The precise form of $\mathbf{F}$ depends on the geometry and thermodynamic conditions: In the case of vibrations of the pinned boundary the main contribution to it comes from the surface tension, while for free motion it should usually be adequate to take it as the difference ΔG_{AB} of the Gibbs free energy per unit volume at the pressure and temperature of the metastable A liquid.[12]

In applying formula (1) let us first consider the experiments of Buchanan, Swift, and Wheatley.[8] We will verify below that at the relevant temperatures the relaxation time M^*/Γ should be of order 10^{-9} s, so that we would expect that in those experiments (time scale $\gtrsim 10^{-2}$ s) the boundary should move at a terminal velocity v_{AB} given by $v_{AB} = \Delta G_{AB}/\Gamma$. Thus it remains only to calculate the friction coefficient Γ.

Let us assume that the boundary is moving (relative to the cell walls) with velocity $\mathbf{v}$ ($\equiv \dot{\mathbf{u}}$) while the A phase some distance ahead of it, say a few mean free paths, is characterized by two-fluid flow with normal and superfluid velocities $\mathbf{v}_n$ and $\mathbf{v}_s$, respectively. We shall initially assume that $\mathbf{v}_n$ and $\mathbf{v}_s$ are both zero (on both A and B sides), and return later to examine this assumption. To calculate the friction coefficient Γ, we assume that the condensed Cooper pairs transform their wave functions adiabatically as the boundary passes and therefore contribute no frictional force (though see below). Then the only mechanism which can produce such a friction is the change of momentum suffered by a normal quasiparticle when reflected from, or transmitted across, the moving boundary. In considering this effect it is essential to note that the time spent by a typical quasiparticle in the boundary region, which is of order d/v_F, is always short compared to a typical collision time [condition (1) above], and hence the relaxation of the transmitted or reflected quasiparticles to equilibrium takes place overwhelmingly in the bulk A or B phase, where the order parameter is not a function of position. Thus in strong contrast to the case of (say) a moving A-phase texture whose characteristic dimension is large compared to

© 1986 The American Physical Society

the mean free path $l(T)$, the concept of orbital viscosity[13] is irrelevant, and the correct calculation simply consists in working out the extra change in the total momentum of the quasiparticle system per unit time due to reflection from, or transmission across, the moving boundary and setting it equal to $\Gamma\mathbf{v}$.

Consider for definiteness a quasiparticle incident on the boundary from the A side with energy E and a wave vector $\mathbf{k}$ of magnitude greater than the Fermi wave vector k_F (E and $\mathbf{k}$ are measured in the rest frame of the superfluid, i.e., that of the cell walls), and in a direction specified by azimuthal and polar angles θ,ϕ, where θ is measured from the inward normal $\hat{\mathbf{n}}$ to the boundary and ϕ from the common plane of $\hat{\mathbf{n}}$ and the characteristic orbital vector[2] $\mathbf{l}$. Since the A-phase gap Δ_A is a function of θ and ϕ, the quasiparticle "kinetic energy" $\epsilon_k \cong \hbar v_F(k-k_F)$ (i.e., the normal-state energy of the state $\mathbf{k}$ relative to the Fermi energy) may be regarded as a function of E, θ, and ϕ:

$$\epsilon_A(E,\theta,\phi) \equiv +(E^2-|\Delta_A(\theta,\phi)|^2)^{1/2}. \quad (2)$$

Suppose for definiteness that E is less than the isotropic B-phase gap Δ_B. Then the quasiparticle will, of course, be reflected from the boundary, not by ordinary but by Andreev[14] reflection. Consider first the case $\mathbf{v}=0$. Then the reflection process must conserve both the component of $\mathbf{k}$ parallel to the boundary and the energy E, so that the quasiparticle simply emerges as a quasihole with kinetic energy $-\epsilon_A$; moreover, provided $\cos\theta$ is not too small ($\lesssim \Delta/\epsilon_F$) the resulting change in its momentum, which we label $\Delta p^{(0)}$, is simply $-(2\epsilon_A/v_F\cos\theta)\hat{\mathbf{n}}$. Such a process, and the obviously related ones (cf. below), give rise to a finite force on the boundary even when $\mathbf{v}=0$; it is straightforward to show[9] that this force is just the quasiparticle contribution to the change in free energy when the A-B boundary is adiabatically displaced, and it should therefore be counted as part of the conservative force $\mathbf{F}$ in Eq. (1).

Now suppose that the boundary is moving with finite velocity $\mathbf{v}$. Now what must be conserved, apart from the parallel component of $\mathbf{k}$, is the energy *in the frame of the moving boundary,* that is the quantity $E' \equiv E-\hbar\mathbf{k}\cdot\mathbf{v}$ ($+$const). Expanding up to first order in $v/(v_F\cos\theta)$, we find that the *extra* momentum change $\Delta p^{(1)}$ over and above that for the stationary boundary is given by the expression

$$\Delta p^{(1)} = [(2E/v_F^2)\sec^2\theta]\mathbf{v}. \quad (3)$$

To find the frictional force due to reflection of A-phase quasiparticles, we must multiply the total momentum change, $\Delta p^{(0)}+\Delta p^{(1)}$, by the flux of quasiparticles incident on the boundary with energy in the range dE and wave vectors in the solid angle $d\Omega$, integrate over energy and (relevant) angle, and keep the term linear in $\mathbf{v}$ [the resulting expression is clearly, for each value of θ, the leading term in an expansion in $v/v_F\cos\theta$)]. The relevant flux is given by the expression

$$(4\pi)^{-1}\frac{dn}{d\epsilon}\frac{E}{\epsilon_A}f(E)\left(\frac{v_F\epsilon_A}{E}\cos\theta+v\right),$$

where $dn/d\epsilon$ is the (normal state) density of states (of both spins) at the Fermi surface, $\epsilon_A(E,\theta,\phi)$ is given by Eq. (2), and $f(E)$ is the Fermi function. Note that we have explicitly assumed, here, that the incident quasiparticle distribution is the thermal equilibrium one in the frame of the walls (cf. below).

To obtain the total friction coefficient Γ we clearly have to generalize the calculation to include (a) quasiholes, (b) quasiparticles and quasiholes incident from the B phase, and (c) the possibility of transmission across the boundary. Omitting the straightforward algebra involved, we quote the final result for Γ:

$$\Gamma = (2/v_F)(dn/d\epsilon)\int(d\Omega/4\pi)|\sec\theta|\int_0^\infty dE\,Ef(E)\{(2-\bar{T})[\theta(E-|\Delta_A(\theta,\phi)|)+\theta(E-\Delta_B)] - \theta(E-\Delta_{\max}(\theta,\phi))\bar{T}[\epsilon_A^2(E,\theta,\phi)+\epsilon_B^2(E)]/\epsilon_A\epsilon_B\}. \quad (4)$$

Here $\theta(x)$ is the usual Heaviside step function, ϵ_B is $+(E^2-\Delta_B^2)^{1/2}$, $\Delta_{\max}(\theta,\phi)$ is the larger of $|\Delta_A(\theta,\phi)|$ and Δ_B, $\bar{T}(E,\theta,\phi)$ is the spin-averaged transmission coefficient, and the angular integral goes over *all* solid angle. Equation (4) is the principal quantitative result of this paper. The right-hand side is clearly logarithmically divergent unless some lower cutoff is put on $|\cos\theta|$. Since the approximations implicit in the calculation fail whenever $|\cos\theta|$ is small compared to any of the quantities Δ/ϵ_F, l/ξ_0, or v/v_F, we should presumably take the cutoff to be of the order of the largest of these. In the numerical calculations quoted below we have taken it to be equal to v/v_F; since the dependence on the cutoff is only logarithmic and the other two quantities are at most of order 5×10^{-3}, while v/v_F is never less[8] than about 5×10^{-4} except possibly for the last data point, the error so incurred is never more than a factor of about 1.4 and is usually much less.

The general order of magnitude of the quantity Γ, for $T\sim T_c$, is easily seen to be $\Delta^2(T)(dn/d\epsilon)v_F^{-1}$; note that this is a factor of order $(\Delta/\epsilon_F)^2$ smaller than the "friction coefficient" we should expect if all the atoms in the A liquid were reflected in the usual (not Andreev) way from the moving boundary. This fact is

crucial in the discussion of a possible finite $\mathbf{v}_n$ and $\mathbf{v}_s$, to which we now turn.

If $\mathbf{v}_n$ is finite, then, since on the relevant time scale the mass current is zero, $\mathbf{v}_n - \mathbf{v}_s$ is also finite; moreover, in general $\mathbf{v}_n$ and $\mathbf{v}_s$ may be different on the two sides of the boundary. Under these conditions the analysis is quite delicate even[9] for $\mathbf{v}_{nA} = \mathbf{v}_{nB}$, and we shall not go into it here, merely remarking that we expect (this contribution to) the friction to vanish when $\mathbf{v}_n = \mathbf{v}$ on both sides of the boundary (even if $\mathbf{v}_{SA} \neq \mathbf{v}_{SB} \neq \mathbf{v}$) and in general to be a function of $\mathbf{v}_n - \mathbf{v}_s$ (as is also the conservative force $\mathbf{F}$) on a scale which is likely to be not the Fermi velocity v_F but the much smaller "pair-breaking critical velocity" $v_c \equiv 2\Delta/p_F \sim (\Delta/\epsilon_F)v_F$. (Indeed, when $|\mathbf{v}_n - \mathbf{v}_s|$ exceeds v_c the very concepts of the two-fluid model may break down.) Thus, *prima facie,* the necessary conditions to apply the results of the above calculation, done for $\mathbf{v}_n = \mathbf{v}_s = 0$, should be (a) $|\mathbf{v}_n| \ll |\mathbf{v}|$, and (b) $|\mathbf{v}_n - \mathbf{v}_s| \ll v_c$. At first sight it seems very unlikely that these conditions, particularly (b), would be fulfilled in a realistic experiment: Even if we exclude the case of any substantial preexisting counterflow in the metastable A liquid (as we shall for present purposes) it appears *a priori* likely that the very motion of the boundary would itself induce a $\mathbf{v}_n$ (hence a $\mathbf{v}_n - \mathbf{v}_s$) of order at least v_c, if not greater, in the liquid ahead of it.

That this argument fails is entirely due to the anomalously small value of the coefficient Γ [Eq. (4)]. Suppose we define a momentum-transfer coefficient K between the bulk phases and the walls by $\dot{\mathbf{P}} \sim K\mathbf{v}_n$, where $\dot{\mathbf{P}}$ is the momentum transferred per unit time between the bulk A and B liquids and the walls and $\mathbf{v}_n$ is a typical value of the normal velocity in the bulk. A straightforward hydrodynamic calculation[9] shows that in a tube of radius R, K is at least of order[15] ηR if $R \lesssim l(T)$ and of order $\eta R^2/l(T)$ in the opposite case, where η is the ordinary normal-fluid viscosity. Thus the ratio of ΓR^2 to K is at most of order $(\Delta/\epsilon_F)^2$, and since these two quantities play the role of two conductances in series it follows that the smaller, ΓR^2, totally dominates and so the bulk normal velocity is only of order $(\Delta/\epsilon_F)^2\mathbf{v} \lesssim (\Delta/\epsilon_F)v_c \ll v_c$. We conclude that provided there was no substantial two-fluid counterflow initially in the A phase, the expression (4) is an excellent approximation to the actual friction on the boundary.

In the limit of low temperature ($k_B T \ll \Delta_B$) we can set $\bar{T} = 0$ in Eq. (4) without appreciable error, and the right-hand side is then also independent of the cutoff provided that θ_0, the angle made by the $\mathbf{l}$ vector with $\hat{\mathbf{n}}$, is not too close to $\pi/2$. In fact, in this limit Γ is just equal to $[2E_A(T)/v_F]|\sec\theta_0|$, where $E_A(T)$ is the thermal energy of the A phase, and thus is proportional to T^4 (though see below). In the region of the experiments of Buchanan, Swift, and Wheatley[8] ($T/T_c \gtrsim 0.6$), explicit evaluation of Eq. (4) requires a knowledge of the transmission coefficient $\bar{T}$. For present purposes we shall make the simple *Ansatz* $\bar{T} = \theta(E - \Delta_{\max}(\theta,\phi))$. With this approximation (which will if anything underestimate Γ and hence overestimate v_{AB}, the error decreasing with distance from T_{AB}), and the experimental values[16] of $\Delta G_{AB}(P,T)$ (with $H = 0$), we obtain the terminal velocity v_{AB} of the boundary as a function of temperature; the comparison with the (lower field) data of Buchanan, Swift, and Wheatley[8] is shown in Fig. 1.[17] The apparently impressive quantitative agreement of the melting-pressure "perpendicular" curve with the 33.6-bar data may be an accident, since under the conditions of the experiment one might *prima facie* expect $\mathbf{l}$ to lie predominantly *across* the tube, i.e., parallel to the boundary (cf. Ref. 2, Sec. X): on the time scale of the experiment there is, of course, no time for $\mathbf{l}$ to adjust (cf. Ref. 13). However, it is clear that both the order of magnitude and the general trend of v_{AB} are in good agreement with the theory. (On the rapid upturn of the 27-bar theoretical curve, see footnote 16.) Note that from the point of view of the theory, the pressure independence of the limiting slope of v_{AB} as a function of $\Delta T/T_{AB}$ may be a numerical accident: In fact for P close to the polycritical pressure P_c, we expect this slope to vary as $P - P_c$.

The theory developed above is appropriate to the temperature regime of the experiment ($T/T_c \gtrsim 0.6$). However, we should expect that at the lowest temperatures the terminal velocity would be limited by a quite different mechanism, namely "pair breaking" (excitation of the Cooper pairs from the "ground pair"[2] to

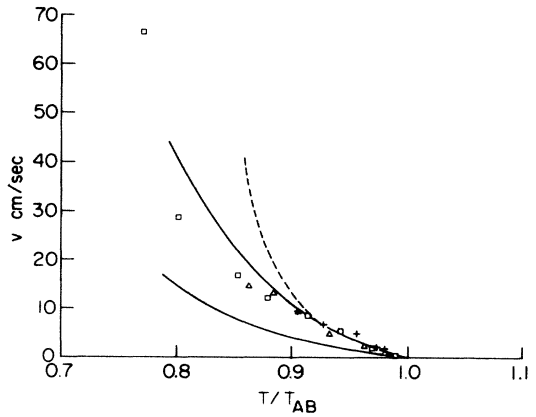

FIG. 1. Velocity of the phase boundary $v \equiv v_{AB}$ as a function of the reduced temperature $T/T_{AB}(P)$. Data points are from Ref. 8. Full curves, theoretical results at melting pressure (upper and lower curves are for $\hat{\mathbf{l}}$ respectively normal and parallel to the plane of the phase boundary). Broken curve, theoretical result at 27 atm with $\hat{\mathbf{l}}$ normal to boundary. All curves are for $H = 0$.

the "excited pair" state) by the time-varying self-consistent pair field of the moving boundary. A straightforward order-of-magnitude calculation[9] using adiabatic perturbation theory gives at $T=0$ (for $v \ll v_F$ and $\hbar\omega \ll \Delta$) a damping $\Gamma(v{:}\omega)\mathbf{v}$, where

$$\Gamma(v{:}\omega) \sim (dn/d\epsilon)(\Delta^2/v_F)(\xi/d)^4(v/v_F)^3$$

for the freely moving boundary (provided $v \gg v_c$ and the superfluid is at rest with respect to the walls) and $\Gamma(v{:}\omega) \sim (dn/d\epsilon)(\Delta^2/v_F)(\hbar\omega/\Delta)^3$ for the pinned oscillating case. Thus, for example, if we define the quantity $\lambda \equiv \Delta G_{AB}(P,0)/(\Delta_B^2\, dn/d\epsilon)$ ($\sim 10^{-2}$ at the melting pressure in zero field), we find that the limiting low-temperature value of the terminal velocity should be of order $\lambda^{1/4}(d/\xi)v_F$ (i.e., possibly comparable to v_F itself), and should be attained at a temperature of order $\lambda^{3/16}(\xi/d)^{1/4}T_c$; we hope to give a more quantitative estimate elsewhere.

We have also estimated the inertial mass of the boundary by assuming that the "kinetic energy" associated with the time variation of the order parameter which its motion induces is of the same order as in a homogeneous situation. In this way we find[9] $M^* \sim \sigma/v_F^2 \sim 10^{-12}$ g/cm^2, where σ is the surface tension. Note that this is only a fraction of order $(\Delta/\epsilon_F)^2$ of the total mass of the liquid in the boundary region; this is not surprising since, when the boundary moves, only a very small fraction of the atoms actually change their state.

With the above estimates for M^* and Γ, it is straightforward to obtain a qualitative idea of the behavior of the boundary in many situations of experimental interest.[9] For example, if the boundary is pinned at low temperatures in an aperture of radius R ($\gg d$), it is easy to see that the fundamental mode will have a frequency of order $v_F R^{-1}$, and will be underdamped provided $(T/T_c)^4 \ll \xi/R$, a condition which for reasonable R, say 100 μm, is probably just within the range of existing cryogenic technology.

We are very grateful to Scott Buchanan, Greg Swift, and the late John Wheatley for many helpful discussions and for keeping us continuously informed of the progress of their experiment. We also enjoyed helpful discussions with Gordon Baym and Chris Pethick. This work was supported by the National Science Foundation under Grant No. DMR 83-15550.

[1]J. C. Wheatley, Rev. Mod. Phys. **47**, 415 (1975) (experimental review).

[2]A. J. Leggett, Rev. Mod. Phys. **47**, 331 (1975) (theoretical review).

[3]D. D. Osheroff and M. C. Cross, Phys. Rev. Lett. **38**, 905 (1977).

[4]M. C. Cross, in *Quantum Fluids and Solids 1977,* edited by S. B. Trickey, E. D. Adams, and J. W. Dufty (Plenum, New York, 1977).

[5]R. Kaul and H. Kleinert, J. Low Temp. Phys. **38**, 539 (1980). Note that this calculation is valid in the Ginzburg-Landau region; the resulting value of d/ξ (~ 2.5) may not be applicable at low temperature.

[6]S. Yip, Phys. Rev. B **32**, 2915 (1985).

[7]The oscillations of the boundary at $T=0$ are discussed by A. V. Markelov, Pis'ma Zh. Eksp. Teor. Fiz. **42**, 151 (1985) [JETP Lett. **42**, 186 (1985)] using concepts apparently quite different from ours. We also note the interesting similarities and differences between the present problem and that of the ^{4}He solid-superfluid phase-boundary dynamics [see, e.g., R. M. Bowley and D. O. Edwards, J. Phys. (Paris) **44**, 723 (1983)].

[8]D. S. Buchanan, G. W. Swift, and J. C. Wheatley, preceding Letter [Phys. Rev. Lett. **57**, 341 (1985)].

[9]S. Yip and A. J. Leggett, unpublished.

[10]Condition (1) fails very close to T_c ($1 - T/T_c \lesssim 10^{-4}$). Condition (2) may be marginal for free expansion of the B phase at very low temperatures; cf. below. Condition (3) would fail, e.g., for pinning in ultrasmall orifices, or high-frequency modes in larger ones.

[11]N. A. Greaves and A. J. Leggett, J. Phys. C **16**, 4383 (1983).

[12]A simple estimate based on the thermal-impedance results of Ref. 6 shows that the boundary acts as an almost perfectly diathermal barrier, except at the lowest temperatures where the temperature dependence of G is negligible anyway. Also, the heating effect due to the latent heat is negligible except for (say) $1 - T/T_{AB} \lesssim 10^{-2}$; see Ref. 8.

[13]D. N. Paulson, R. L. Kleinberg, and J. C. Wheatley, in *Proceedings of the Fourteenth International Conference on Low Temperature Physics,* edited by M. Krusius and M. Vuorio (North-Holland, Amsterdam, 1975), Vol. 5, p. 417; M. C. Cross and P. W. Anderson, *ibid.*, Vol. 1, p. 29.

[14]A. F. Andreev, Zh. Eksp. Teor. Fiz. **46**, 1823 (1964) [Sov. Phys. JETP **19**, 1228 (1964)].

[15]This estimate fails at very low temperatures in very wide tubes, but by that time other mechanisms dominate the friction anyway; cf. below.

[16]At melting pressure we used the data of W. P. Halperin *et al.*, Phys. Rev. B **13**, 2124 (1976); at 27 atm, the data given in Fig. 16 of Ref. 1. Note that extrapolation of the latter almost certainly overestimates ΔG_{AB} and hence v_{AB} for T appreciably below T_{AB}.

[17]Note that at general temperatures there is no reason for Γ to be a monotonic function of θ, even in the indicated approximation; indeed, an analytic calculation (Ref. 9) in the limit $T \to T_{AB} \to T_c$ shows that in this limit the value of Γ at $\theta_0 = \pi/2$ is nearly equal to that at $\theta_0 = 0$, but the value at $\sin^{-1}3^{-1/2}$ is about half the latter. This feature, and more generally the strong anisotropy of Γ, raise the intriguing possibility that the boundary may spontaneously roughen and then propagate by a "tacking" mechanism.