Order, Disorder, and Phase Turbulence

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Phase turbulence—a phenomenon associated with the time evolution of extended spatial patterns—is investigated on the basis of the Kuramoto-Sivashinsky equation. The turbulent behavior may be thought of as disorder of a primary cellular pattern and described by hydrodynamic analysis of slow modes. Such an approach explains qualitatively the spatial fluctuation spectrum. It also suggests that the ordering of the system into a linearly stable cellular state is inhibited and the lifetime of the turbulent state grows exponentially with the size of the system. This anomalous glasslike relaxation was observed in a numerical simulation.

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The formation and evolution of dynamical structures and patterns is one of the most exciting and mysterious areas of nonlinear phenomenology. Some of the more familiar experimental examples include the formation of periodic spatial patterns in Rayleigh-Bénard convection or directional solidification. We are primarily interested in the large–aspect-ratio (large number of cells) phenomena,[1] as opposed to small–aspect-ratio systems, which may be adequately understood as dynamical systems with a few degrees of freedom.[2] The former case poses a number of difficult questions concerning the appearance of low-frequency noise and transition to turbulence. Many qualitative features of the systems mentioned above are exhibited in the temporal behavior of one-dimensional structures described by the Kuramoto-Sivashinsky (KS) equation. The KS equation was derived by Kuramoto[3] to describe the evolution of propagating patterns in chemical reaction-diffusion systems and by Sivashinsky in the study of cellular flames.[4] We shall write it here in the scaled form:

$$u_t + u_{xx} + u_{xxxx} + uu_x = 0, \tag{1}$$

where $u_t \equiv \partial u/\partial t$, $u_x \equiv \partial u/\partial x$, and $x \in [0,L]$ with periodic boundary conditions. The system size L will serve as a control parameter. The Kuramoto-Sivashinsky equation is known to generate intrinsic stochasticity, which has been investigated in some detail.[5–7] For large enough values of L, the KS equation generates low-frequency noise[5] and long-wavelength fluctuations.[5,6] On the other hand, it also describes cellular patterns, which have been observed in windows of the control parameter.

The nontrivial behavior of the KS equation stems from the linear instabilitiy of the laminar, $u = \text{const}$, solution. The rate of growth s of a linear mode with wave number k is $s = k^2 - k^4$, so that the long-wavelength modes are unstable. (We are interested in the large-system limit, $L >> 1$.) The instability term u_{xx} balances the dissipation u_{xxxx} on a length scale of approximately 1: Thus one expects there to be relevant structures on that length scale. Indeed, in a numerical simulation, an evolving pattern (Fig. 1) exhibits obvious cellular structure. The pattern is distorted gently, save for the defects corresponding to creation or annihilation of basic cells (the space-time dislocations). This observation suggests a hydrodynamic description of the disorder pattern focusing on the "slow" long-wavelength distortions of stationary periodic solutions. Such an approach has been used in many applications, in particular, in the study of the Rayleigh-Bénard convection[7,8] and, most recently, in the linear-stability analysis of the KS equation by Frisch, Che, and Thual.[9] Below, by means of symmetry arguments, I shall derive a set of *nonlinear* "soft mode" equations. Besides the linear viscoelastic behavior found by Frisch, Che, and Thual,[9] these equations expose a finite-amplitude instability of the regular pattern leading to the formation of shocks in the phase variable, which can be interpreted as space-time dislocations.

On the basis of the hydrodynamic analysis this paper proposes that the phase turbulence can be described as a dynamical equilibrium of "dislocation" events and viscoelastic phase modes. This simple picture qualita-

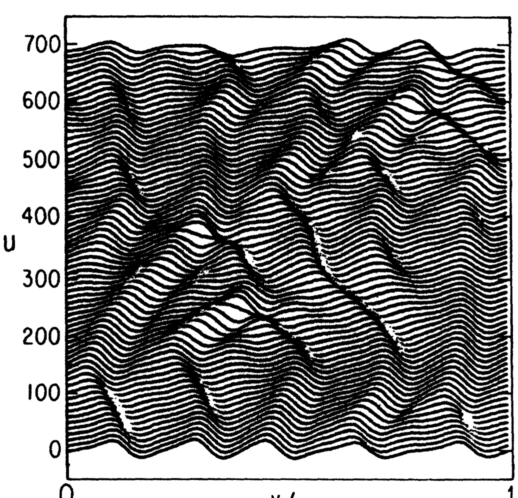

FIG. 1. A sequence of snapshots of the time evolution governed by the KS equation for $L = 2\pi \times 9.25$ taken at $\Delta t \approx 1$ intervals (256 Fourier modes, time step of 0.016).

tively explains the observed long-wavelength fluctuation spectrum, predicts its dependence on L, and explains why the turbulent state does not rapidly decay into a regular pattern, even when such a pattern is linearly stable. The ordering is inhibited by propagation of disturbances and requires a macroscopic fluctuation. The phase-turbulent state has an anomalously long lifetime, increasing exponentially with the size of the system. This conclusion is confirmed by a numerical study of "turbulent lifetimes" for a small-sized system.

The possibility of similar glasslike behavior was suggested by Siggia and Zippelius[8] in the context of the Rayleigh-Bénard convection with free-slip boundary conditions. The similarity of the two problems is due to the fact that both of them possess, in addition to translational invariance, an extra continuous symmetry—Galilean invariance. The latter, as was recently emphasized by Coullet and Fauve,[10] is responsible for converting a diffusive phase mode[7] into a propagating one. In realistic systems the Galilean invariance is likely to be broken, i.e., by rigid boundary conditions. The effect of weak symmetry breaking may be studied in the KS equation with the conclusion that the propagative nature of the phase mode at finite (but small) wave number persists. Thus the qualitative features of phase turbulence may be observable in a broader class of systems.

We now proceed with the theory. The stationary periodic[11] solutions of the KS equation $\overline{U}_\lambda(x)$ with wavelength λ can be found by solving the equation

$$u_{xxx} + u_x + u^2/2 - \mu^2/2 = 0, \tag{2}$$

obtained from Eq. (1) by dropping the time derivative and integrating once. Assume that the average of $\overline{U}_\lambda$ over the period, $\langle \overline{U}_\lambda \rangle$, vanishes and $\overline{U}_\lambda(x)$ is odd. Equation (2) can be solved perturbatively for $\mu^2 = \langle \overline{U}_\lambda^2 \rangle \ll 1$ or numerically.[12] The result is a continuous family of cellular solutions which can be parametrized by wavelength $\lambda/2\pi \geqslant 1$.

Let us pick a particular pattern $\overline{U}_\lambda(x)$ with period $\lambda \sim 1$ and study its evolution for large L. The most relevant distortions are identified by considering the continuous symmetries of Eq. (1): translation, $x \to x + c$, $u \to u$, and the Galilean transformation, $x \to x - vt$, $u \to u + v$. The soft modes are then the slowly varying phase $\phi(x,t)$, which is related to the translation, and an *additive* mode $\xi(x,t)$, representing a weakly nonuniform Galilean transformation:

$$u(x,t) = \overline{U}_\lambda(x + \phi(x,t)) + \xi(x,t) + \chi(x;\phi,\xi) \tag{3}$$

(χ is a "shape" correction depending on the modes ϕ, ξ). The Galilean transformation involves a change in phase, so that ξ is of the same order as $\partial_t\phi$ and $\partial_x\phi \sim \epsilon \ll 1$. The standard multiple-scale analysis methods[13] allow one to derive equations describing the evolution of the soft modes.

The general form of these equations can be obtained by further exploiting symmetry arguments.[14] Note that Eq. (1) possesses a reflection symmetry: $x \to -x$, $u \to -u$. Under reflection, translation, and Galilean transformations ϕ behaves like x, while ξ behaves like u. We then seek equations for $\partial_t\phi$ and $\partial_t\xi$ respecting these symmetries in the form of polynomials in ϕ_x, ξ, and their derivatives. To the third order one gets[15]

$$\phi_t + \xi + \xi\phi_x + \gamma\xi_{xx} + \alpha\phi_{xx} + \beta_\alpha\phi_x\phi_{xx} = 0, \tag{4a}$$

$$\xi_t + \xi\xi_x + \nu\xi_{xx} + \sigma\phi_{xx} + \beta_\sigma\phi_x\phi_{xx} = 0. \tag{4b}$$

The coefficients α, σ, γ, and ν are obtained by multiple-scale analysis[16] and are functionals of the primary pattern $\overline{U}_\lambda(x)$ and therefore implicit functions of the primary wavelength λ. The phase dynamics is independent of the choice of the reference state. This is assured by a relation between linear and nonlinear terms similar to the Ward identities:

$$\begin{aligned} &\lambda\, d\alpha/d\lambda = -\beta_\alpha(\lambda), \\ &\lambda\, d\sigma/d\lambda = -\sigma(\lambda) - \beta_\sigma(\lambda). \end{aligned} \tag{5}$$

These relations are derived by noting that reference patterns with different wavelengths are related by global dilation $x \to x + \delta k\, x$ (which corresponds to a marginal phase mode $\phi = \delta k\, x$) and investigating the behavior of Eq. (4) under this transformation.

Since the existence of the Galilean symmetry has played an essential role, it is important to understand the effect of the breaking of this symmetry. This can be done by investigating the effect of an additional linear term κu in Eq. (1). Weak symmetry breaking ($\kappa \ll 1$) introduces weak damping of the "Galilean" mode, ξ. It can be analyzed in the same framework[16] and leads to the appearance of a linear term $\kappa\xi$ in Eq. (4b)!

Upon linearizing the soft-mode equations

$$\frac{\partial}{\partial t}\begin{pmatrix}\phi \\ \xi\end{pmatrix} = \begin{pmatrix} -\alpha\partial_x^2 & -1-\gamma\partial_x^2 \\ -\sigma\partial_x^2 & -\kappa-\nu\partial_x^2 \end{pmatrix}\begin{pmatrix}\phi \\ \xi\end{pmatrix}, \tag{6}$$

one obtains a dispersion relation for the growth rate s of a linear mode with wave number k. In the presence of a symmetry-breaking term ($\kappa \neq 0$) there are two regimes. At very long wavelength, $k \ll \kappa$, the effect of the symmetry breaking is essential and the soft modes are overdampled (diffusive),[7] $s_1 = \alpha k^2$, $s_2 = -\kappa$. However, at finite wavelength $\kappa \ll k \ll 1$ the dispersion relation is $s = (\alpha + \nu)k^2 \pm i\sqrt{\sigma}|k|$. For $\tilde{\alpha}(\lambda) \equiv \alpha(\lambda) + \nu(\lambda) < 0$ and $\sigma(\lambda) > 0$ the phase mode is viscoelastic and the primary pattern is linearly stable. This dispersion relation is in agreement with the analysis of Frisch, Che, and Thual,[9] who have found (for $\kappa = 0$) that stability conditions are satisifed for $\lambda/2\pi \in [1.2, 1.3]$ so that linearly stable patterns of n cells exist in the windows of $L \in [n\lambda_{\min}, n\lambda_{\max}]$.

The linearly stable states, however, are unstable to finite-amplitude perturbations involving dilation, since they effectively change the wavelength of the pattern. Consider a dilation of a region l ($1 << l << L$): $\phi = \delta k\, x$ for x in the region l, decaying to 0 outside. For $\delta k > -\beta_\alpha^{-1}\tilde{\alpha}(\lambda)$ the nonlinear diffusivity, β_α, term will cause the local fluctuations to see an effective $\tilde{\alpha}' = \tilde{\alpha}(\lambda) + \beta_\alpha \delta k > 0$ and will lead to an instability. Similarly, compressions will lead to an instability due to negative effective $\sigma' = \sigma + \beta_\sigma \delta k$.

The evolution of the modes beyond the instability is determined by the nonlinear terms in Eqs. (4a) and (4b). One can argue (and demonstrate by a direct numerical simulation[16]) that distortions beyond the instability threshold lead to formation of shocks in the phase variable. While Eqs. (4a) and (4b) were derived on the assumption that the variation of the phase is slow, we observe that, physically, discontinuous solutions for the phase $\phi(x)$ are admissible, as long as the jump in phase corresponds to an insertion or disappearance of a cell: $\phi(x^-) - \phi(x^+) = \pm\lambda$. Thus the formation of a shock corresponds to a space-time dislocation. On a length scale much larger than the size of a cell such an event can be represented as a delta-function source of strain ϕ_x in Eq. (4a) causing the emission of a pair of compression ($\phi_x > 0$) or rarefaction pulses propagating away from the dislocation with velocity $\sqrt{\sigma}$. The interaction of such pulses in turn may lead to creation or annihilation of a cell.

I propose that phase turbulence may be thought of as a dynamic equilibrium of space-time dislocations interacting with "viscoelastic" waves. Let us characterize the distortions of the pattern by an elastic strain energy density

$$E = L^{-1} \int_0^L dx (\xi^2 + \sigma \phi_x^2). \tag{7}$$

This energy is dissipated by "viscous" effects as long as $\alpha + \nu < 0$, which is why the ordered state has a finite domain of attraction. However, E is not conserved in nonlinear processes such as formation or annihilation of cells. If elastic strain exceeds the instability threshold, it will cause dislocation events to occur, increasing E. I conjecture that the system then finds a disordered dynamic equilibrium state characterized by a finite elastic energy density[17] and certain frequency of dislocation events. Even without a more detailed theory this simple statistical point of view has some important implications, which we explore next.

The elastic energy is directly related to the low-k part of the spatial fluctuation spectrum $S(k) \equiv |\tilde{u}(k)|^2$, which is proportional to $|\tilde{\xi}(k)|^2$ and $|k\tilde{\phi}(k)|^2$. Some of the elastic energy is in the form of pulses emitted by dislocation events. Since the Fourier transform of such deltalike pulses is flat, they contribute a constant background to the fluctuation spectrum $S(k \to 0) = S_0$ with S_0 proportional to the number of dislocation events per unit time. The latter (neglecting the correlations) is proportional to the size of the system L and the frequency of dislocation events. This explains the flat shoulder in the fluctuation spectrum[6] and predicts that S_0 increases linearly with L.

Another consequence of this picture is that the turbulent state is long lived. A transition to the ordered state requires a reduction of the *total*[18] elastic energy in order to make the probability of a dislocation event small and have the viscosity dissipate the strain. That implies a *macroscopic* fluctuation and means that the lifetime of the phase-turbulent state should increase exponentially with the size of the system L.

These qualitative predictions can be confirmed by a numerical simulation. The KS equation was simulated on an FPS-164 array processor by use of the pseudospectral method and an Adams-Bashforth integration algorithm. I measured the time-averaged spatial fluctuation spectrum $|\tilde{u}(k)|^2$ for $L/2\pi$ ranging from 13 to 70 and found that $S_0(L)$ is well fitted by a linear function in agreement with the theoretical argument. I also investigated the dependence of the turbulent lifetime, τ, on L for small-size systems with L lying in the windows of stability of n-cellular solutions[19] ($n = 4, \ldots, 10$; $L = 2\pi n \times 1.29$). The results for $\ln(\tau)$ averaged over ensembles of initial conditions are presented in Fig. 2. (The single-run lifetimes fall into a rather broad distribution). A factor of 2 increase in L leads to an increase in lifetime by more than two orders of magnitude. The data suggest that

$$\tau = \tau_0 \exp(L/L_0)^\gamma \tag{8}$$

with $\gamma \approx 1$ [other constants in Eq. (8) depend on the

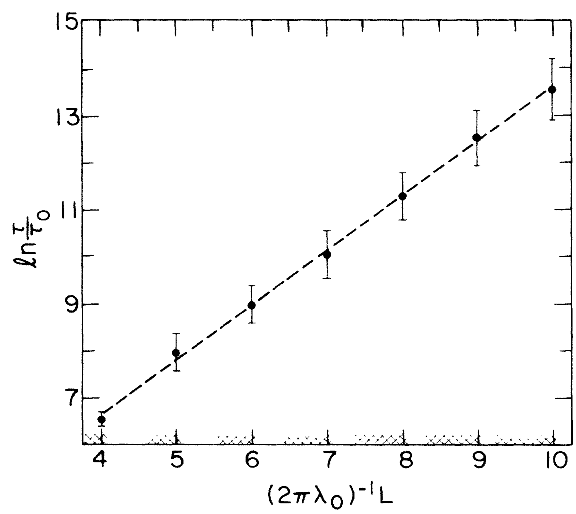

FIG. 2. Logarithm of the lifetime of the phase-turbulent state averaged over ensembles of initial conditions as a function of system size L ($L = 2\pi n \times 1.29$). Shaded are the windows of linear stability of the cellular states.

choice of L within the window], which is a clear indication of the expected anomalously slow relaxation.

In conclusion, the simple hydrodynamic approach outlined above successfully captures many of the features of the chaotic behavior exhibited by the KS equation. This is surprising, since the detailed dynamics of the KS equation with finite L exhibits all the complexity associated with many degrees of freedom[20] (e.g., multiple attractors and sensitive dependence on the initial conditions). However, the results of the present paper suggest that in the statistical sense the behavior of large systems ($L \to \infty$) may allow a simpler description.

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[15]The form of Eqs. (4a) and (4b) can be altered by a "gauge" transformation of the form $\phi \to \phi + a\xi_{xx} + \ldots$ and $\xi \to \xi + b\phi_{xx} + \ldots$. Such a change of variables can be used to eliminate the $\gamma\xi_{xx}$ term in Eq. (4a). Similar equations were obtained independently by P. Coullet and S. Fauve, in Ref. 9.

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[17]Since the spatial fluctuations are equipartitioned among the long-wavelength modes (with Gaussian statistics) (Ref. 6) one can introduce an effective noise temperature proportional to the elastic energy density. The average value of E is determined by the energy balance in the dislocation processes.

[18]Because the disturbances propagate, the ordering must occur simultaneously on the macroscopic scale.

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