Solution to the U(1) Problem on a Lattice

Sinya Aoki
Department of Physics, University of Tokyo, Hongo, Bunkyo-ku, Tokyo 113, Japan, and
Research Institute for Fundamental Physics, Kyoto University, Sakyo-ku, Kyoto 606, Japan
(Received 25 July 1986)

A new resolution to the U(1) problem is proposed without direct reference to spontaneous chiral-symmetry breaking. The parity and flavor symmetry which correspond to the neutral-pion operator are spontaneously broken when we change the mass parameter M. Therefore the neutral pion becomes massless at the phase-transition point where the correlation length diverges, while the η meson remains massive. Furthermore, the charged pion becomes the Nambu-Goldstone boson in the parity- and flavor-symmetry-breaking phase.

PACS numbers: 12.38.Gc

It is important to analyze the chiral-symmetry breaking and the U(1) problem by a nonperturbative method: lattice QCD.[1] But from the beginning we face a difficulty: Chiral symmetry cannot be well defined on a lattice.[2] One way to define the lattice fermion is to use the Wilson fermion formulation which breaks chiral symmetry explicitly. One expects that the chiral symmetry broken by the Wilson term will be restored in the continuum limit. This is, however, a nontrivial expectation. The effective potential of the lattice Gross-Neveu model with one bare coupling constant (that of the four-fermion interaction) does not become chirally symmetric even in the continuum limit.[3] The lattice CP^n model does not satisfy the PCAC (partial conservation of axial-vector current) relation in the continuum limit.[4] We are afraid that chiral symmetry for lattice QCD may not be restored in the continuum limit.

The strong-coupling expansion[5] or Monte Carlo simulation[6] shows that the mass parameter M ($=m_B a+4$; m_B is the bare quark mass) can be chosen so as to make the pion massless. Why does the pion become massless without the use of chiral symmetry? The answer to this question for the one-flavor case was considered by Aoki.[7,8] There are two phases; one is the parity-conserving phase ($\langle\bar\psi i\gamma_5\psi\rangle=0$) and the other is the parity-nonconserving phase ($\langle\bar\psi i\gamma_5\psi\rangle\neq 0$). Since the pion corresponds to the operator $\bar\psi i\gamma_5\psi$, it becomes massless at the second-order phase-transition point.

For the many-flavor case we must not only explain the masslessness of the pion but also solve the U(1) problem at the same time, without using chiral symmetry explicitly. In a previous paper[9] I calculated m_η (η-meson mass) and m_π (π-meson mass) in the strong-coupling expansion of lattice QCD and obtained

$$m_\eta > m_\pi \text{ and } \lim_{m_\pi\to 0} m_\eta \neq 0,$$

in a phase where $\langle\bar\psi i\gamma_5\psi\rangle=0$. The underlying dynamics which gives the mass difference was briefly mentioned there. In this Letter I investigate this point in detail to clarify the mechanism. The formulas, notations, and some of the results are found in Ref. 9. The details of the calculations will be published in a forthcoming paper.[10]

I mainly treat the two-flavor case. The extension to the case of an even number of flavors is straightforward, while extension to an odd number of flavors causes problems (except one flavor). This point will be mentioned in the end of this Letter.

I first summarize the solution to the smallness of the pion mass and the U(1) problem for the two flavors.

(i) There is a second-order phase transition whose order parameter is $\langle\bar\psi i\gamma_5\tau^3\psi\rangle$ when we change the mass parameter M with g^2 fixed:

$$\langle\bar\psi i\gamma_5\tau^3\psi\rangle=0 \text{ for } M^2\geq M_c^2,$$

and

$$\langle\bar\psi i\gamma_5\tau^3\psi\rangle\neq 0 \text{ for } M^2< M_c^2,$$

where $M_c^2=M_c^2(1/g^2)$ is a critical point.[8,9] The parity (at the same time, CP) and flavor symmetry are spontaneously broken for $M^2<M_c^2$. The corresponding Nambu-Goldstone (NG) bosons are the charged pions. Note that this vacuum expectation value is not the standard one.

(ii) The particle corresponding to the order parameter $\bar\psi i\gamma_5\tau^3\psi$, the neutral pion, becomes massless at $M=M_c$. In other words the correlation length $\xi=1/m_{\pi^0}a$ diverges at $M=M_c$, where m_{π^0} is the neutral-π-meson mass. Furthermore $m_{\pi^0}^2>0$ except $M=M_c$. Therefore π^0 is the massless mode associated with this phase transition rather than an NG boson.

(iii) Since the η (flavor singlet) meson is not mixed with other mesons because of the flavor symmetry, $m_\eta\neq m_{\pi^0}$ in general. Furthermore, since $\langle\bar\psi i\gamma_5\cdot 1\psi\rangle$ is always equal to zero, the particle corresponding to $\bar\psi i\gamma_5\cdot 1\psi$, the η meson, stays massive. Therefore, $m_\eta^2>m_{\pi^0}^2$ near the critical point M_c. I thus claim that the third component of the "parity-flavor symmetry" plays an important role in the solution of the U(1) problem. Indeed $m_\eta^2 > m_{\pi^0}^2$ for all $M^2 \geq M_c^2$ in the strong-coupling expansion.

(iv) Since the flavor symmetry is conserved,

$$m_{\pi^0} = m_{\pi^\pm} \text{ for } M^2 \geq M_c^2.$$

The above results (i)–(iv) are compactly illustrated in Fig. 1.

I try to clarify the situation by using the σ model. The field $\bar{\psi} i \gamma_5 \tau^3 \psi$ (π^0 meson) corresponds to the σ field and $\bar{\psi} i \gamma_5 \tau^\pm \psi$ ($\pi^\pm$ meson) corresponds to the π field. The σ field becomes massless at the point where the potential changes to a double-well potential from a single-well potential. The $\bar{\psi} i \gamma_5 \cdot 1 \psi$ (η meson) is a field unrelated to the σ model; therefore there is no reason that this becomes massless.[11]

Now I sketch the analysis. U(N) gauge is used for color group. I take the Wilson fermion formulation with equal mass for each flavor. Therefore the symmetry for the fermion action is $Z_2 \otimes \mathrm{U}(n_f) = (\text{parity}) \otimes (\text{flavor symmetry})$. Both $1/N$ and $1/g^2N$ expansions are used to calculate the effective action for mesons, $S_{\text{eff}}(1/g^2N, 1/N, \phi(x))$, where $\phi(x)$ is the meson field. I divide S_{eff} into two parts:

$$S_{\text{eff}} = S_{\text{eff}}^c + S_{\text{eff}}^s,$$

where S_{eff}^c is a common part for both singlet and nonsinglet mesons and S_{eff}^s is the singlet meson part to give the mass difference. For simplicity I use the effective potential obtained in the large-N and strong-coupling limit[7,12]

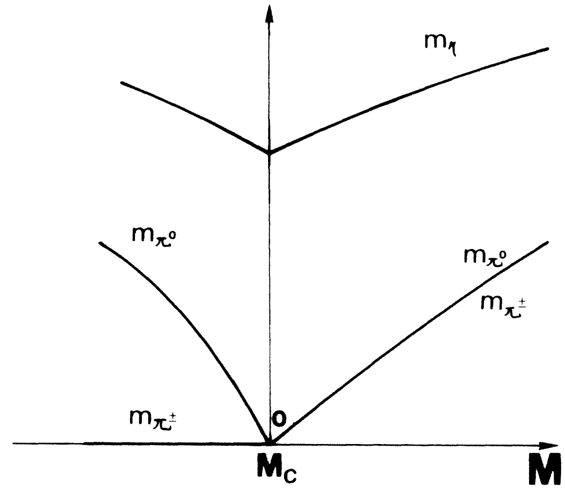

FIG. 1. Dependences of m_η, m_{π^0}, and $m_{\pi^\pm}$, on M with g^2 fixed. Note that $\langle \bar{\psi} i \gamma_5 \tau^3 \psi \rangle = 0$ for $M \geq M_c$ and $\langle \bar{\psi} i \gamma_5 \tau^3 \psi \rangle \neq 0$ for $M < M_c$.

for S_{eff}^c:

$$S_{\text{eff}}^c = \sum_n \{ \mathrm{tr} M \phi(n) - \tfrac{1}{2} \ln \phi(n)^2 + \sum_\mu \omega(\Lambda n, \mu) \}, \quad (1)$$

where

$$w(x) = \ln[1 + (1+4x)^{1/2}] - (1+4x)^{1/2},$$

$$\Lambda_{n,\mu} = \phi(n)(P_\mu)^t \phi(n+\mu)(P_{-\mu})^t \text{ and } P_\mu = \frac{1 \pm \gamma\mu}{2}.$$

The leading term in the strong-coupling expansion for S_{eff}^s has been calculated in Ref. 9,

$$S_{\text{eff}}^s = \mathrm{const}(1/g^2N)^6 (1/N)^5 \left(\sum_{\mu_1+\mu_2+\cdots+\mu_8=0} \mathrm{tr}\phi(n)(P_{\mu_1})^t \phi(n+\mu_1)(P_{\mu_2})^t \cdots \phi(\mu_1 + \ldots + \mu_n)(P_{\mu_8})^t \right)^2. \quad (2)$$

Next we calculate the vacuum expectation value and show that the phase transition occurs. Let us assume that the vacuum expectation value has the following form:

$$\langle \phi(n)_{\alpha\beta}^{ij} \rangle \equiv \langle \bar{\psi}(n)_\alpha^i \psi(n)_\beta^j \rangle = \sigma_i \delta_{ij} (\exp i\theta_j \gamma_5)_{\alpha\beta}, \quad (3)$$

where i,j are the flavor indices and α,β are the spinor indices. We insert (3) into S_{eff}:

$$V_{\text{eff}} = -S_{\text{eff}}/4NV = \sum_{i=1}^{2} [\tfrac{1}{2} \ln \sigma_i^2 - M\sigma_i \cos\theta_i + 2(1 - 4\sigma_i^2 \sin^2\theta_i)^{1/2} - 2\ln\{1 + (1 - 4\sigma_i^2 \sin^2\theta_i)^{1/2}\}]$$
$$+ 3b \left(\sum_{i=1}^{2} \sigma_i^8 (\sin^4\theta_i + 2\sin 2\theta_i) \right)^2 - fb \left(\sum_{i=1}^{2} \sigma_i^8 (1 - \cos 4\theta_i \sin^2 2\theta_i) \right)^2, \quad (4)$$

where V is the volume, $b = \frac{3}{16}(1/g^2N)^6/N^5$, and f is the number of the diagrams which contribute to the last term of (4). Since the last term is irrelevant for the phase transition we omit it hereafter. Gap equations derived from (4) become

$$M\sigma_i = \cos\theta_i \quad (i = 1,2),$$
$$-\sin\theta_i + \frac{8\sigma_i^2 \sin\theta_i}{1 + (1 - 4\sigma_i^2 \sin^2\theta_i)^{1/2}} - 24b\sigma_i^8 (\cos 4\theta_i + \cos 2\theta_i) \left\{ \sum_{j=1}^{2} \sigma_j^8 (\sin 4\theta_j + 2\sin 2\theta_j) \right\} = 0. \quad (5)$$

There are two solutions to (5):

(a) $\theta_1 = -\theta_2$ or (b) $\theta_1 = \theta_2$;

solution (a) has lower energy than that of (b). Indeed solution (a) has a critical point $M_c^2 = 4$ and solution (b) has another critical point $M_c^2 = 4 - 3b/64$. Solution (a) has the same form as that of the strong-coupling limit for one flavor[7,8]:

$$\sigma_1 = \sigma_2 = \begin{cases} 1/M & \text{for } M^2 \geq M_c^2 = 4, \\ [3/(16-M^2)]^{1/2} & \text{for } M^2 < M_c^2, \end{cases}$$

$$\sin\theta_1 = -\sin\theta_2 = \begin{cases} 0 & \text{for } M^2 \geq M_c^2, \\ \dfrac{2(4-M^2)^{1/2}}{(16-M^2)^{1/2}} & \text{for } M^2 < M_c^2. \end{cases}$$

This solution means

$$\left.\begin{aligned} &\langle \bar{\psi} i\gamma_5 \tau^3 \psi \rangle = 2\sigma_1 \sin\theta_1 \neq 0, \\ &\langle \bar{\psi} i\gamma_5 \cdot 1 \psi \rangle = 0, \end{aligned}\right\} \text{ for } M^2 < M_c^2.$$

In other words the spontaneous breakdown of parity (CP) and flavor symmetry occurs for $M^2 < M_c^2$. The operator $\bar{\psi} i\gamma_5 \tau^3 \psi$ corresponds to the neutral pion which becomes massless at $M = M_c$. Indeed for $M^2 > M_c^2 = 4$

$$\cosh(m_{\pi^0} a) = \cosh(m_{\pi^\pm} a) = 1 + \frac{(M^2-4)(M^2-1)}{2M^2-3},$$

while

$$\cosh(m_\eta a) = 1 + \frac{(M^2-4+96t)(M^2-1-22t)}{(1-37t)(2M^2-3)},$$

$$t = \frac{b}{2048}.$$

At $M = M_c$, $m_{\pi^0}^2 = m_{\pi^\pm}^2 = 0$ and $m_\eta^2 > 0$. For $M^2 < M_c^2$ the NG boson must appear because the flavor symmetry is broken:

$$\cosh(m_{\pi^\pm} a) = 1.$$

Since the neutral pion is not an NG boson it becomes massive for $M^2 < M_c^2$,

$$\cosh(m_{\pi^0} a) = 1 + \frac{2(4-M^2)(16-M^2)(8+M^2)}{15M^4 - 64M^2 + 256}.$$

The η-meson mass for $M^2 < M_c^2$ is very complicated. Therefore I do not calculate it here. Instead I consider a simple model which clarifies the mechanism,

$$S_{\text{eff}}^s = (b/4) \sum_n \{\operatorname{tr} \gamma_5 \phi(n)\}^2.$$

I calculate the η-meson mass from this effective action and S_{eff}^c in (1) and obtain

$$\cosh(m_\eta a) = 1 + \frac{(M^2-4+4b)(M^2-1)}{2M^2-3+4b} \quad \text{for } M^2 > M_c^2 = 4,$$

$$\cosh(m_\eta a) = 1 + 2(8+M^2) \frac{(4-M^2)(16-M^2)^2 + 2b(7M^2-16)(8+M^2)}{(16-M^2)(15^4 - 64M^2 + 256) + 16b(8+M^2)(7M^2-16)} \quad \text{for } M^2 < M_c^2.$$

The η meson stays massive for all M.

The essential point of this solution to the U(1) problem is summarized as follows. The original action has the symmetry (parity)$\otimes$(flavor symmetry), while chiral symmetry is lost. The parity and flavor symmetry are spontaneously broken. The order parameter $\langle \bar{\psi} i\gamma_5 \tau^3 \psi \rangle$ varies from zero to nonzero while $\langle \bar{\psi} i\gamma_5 \cdot 1 \psi \rangle$ stays vanishing when we change the parameter M with g^2 fixed. This phase transition makes the π^0 meson massless at the critical point ($M = M_c$) while the η meson remains massive since the order parameter is $\langle \bar{\psi} i\gamma_5 \tau^3 \psi \rangle$ rather than $\langle \bar{\psi} i\gamma_5 \cdot 1 \psi \rangle$. Flavor symmetry also makes the $\pi^\pm$ massless at $M = M_c$ and it remains massless just like a NG boson associated with the flavor-symmetry breaking for $M^2 < M_c^2$.

I finally mention some remaining problems. For $n_f = 1$, parity is spontaneously broken[7] for $M^2 < 4$ and the vacuum has Z_2 degeneracy since $\langle \bar{\psi} i\gamma_5 \psi \rangle = \pm \sigma \sin\theta$. For general n_f the vacuum has $(Z_2)^{n_f}$ degeneracy if we omit the singlet part S_{eff}^s (or $b = 0$). Introduction of the singlet part S_{eff}^s (or $b \neq 0$) resolves the degeneracy and the lowest-energy states of the $(Z_2)^{n_f}$ degenerate states become true vacua. The condition for the lowest-energy states is given by

$$\sum_{i=1}^{n_f} \sigma_i^8 (\sin 4\theta_i + 2\sin 2\theta_i) = 0 \tag{6}$$

in the first order of b. For $n_f = 2n =$ even it is easy to satisfy the above condition as in the case of $n_f = 2$. But for $n_f =$ odd we cannot obtain a reasonable solution to (6). For example,

$$\theta_1 = -\theta_2, \ \theta_3 = 0$$

is the solution for $n_f = 3$, but this solution is problematic since there appears a tachyon corresponding to $\bar{\psi}^3 i\gamma_5 \psi^3$ for small b. Therefore there must be another stable vacuum which does not satisfy the condition (6). It is difficult to find such a solution. I will try to solve this problem in the future.

Besides the strong-coupling expansion, the Monte Carlo (MC) simulations may be useful to confirm the existence of the phase transition I have mentioned. For $n_f = 1$ such a MC simulation is now under investigation in the quenched approximation.[13] For the many-flavor case the analysis in this Letter suggests that the effect of the dynamical quark loops is important to generate the mass difference and to cause the phase transition which I have mentioned. Therefore, careful calculation for the quark determinant of each flavor is necessary to analyze this phase transition in the MC simulation. In Ref. 7 I pointed out that the complicated phase structure may exist in the weak-coupling region. It is very interesting to analyze the phase structure in the weak-coupling region even for the one-flavor case.

I would like to thank Dr. N. Kawamoto for a careful reading of the manuscript and useful discussions. I also thank the Research Institute for Fundamental Physics (Yukawa Hall), Kyoto for its hospitality during part of this research, and particularly Professor M. Fukugita for useful discussions.

[1]K. Wilson, Phys. Rev. D **10**, 2445 (1974).

[2]H. B. Nielsen and M. Ninomiya, Nucl. Phys. **B185**, 20 (1981).

[3]S. Aoki and K. Higashijima, Prog. Theor. Phys. **76**, 521 (1986).

[4]E. Abdalla, M. C. B. Abdalla, and N. Kawamoto, Phys. Rev. D **31**, 3213 (1985).

[5]N. Kawamoto, Nucl. Phys. **B190 [FS3]**, 617 (1981).

[6]For example, see M. Fukugita, T. Kaneko, and A. Ukawa, Phys. Lett. **130B**, 199 (1983).

[7]S. Aoki, Phys. Rev. D **30**, 2653 (1984).

[8]S. Aoki, Phys. Rev. D **33**, 2339 (1984).

[9]S. Aoki, Phys. Rev. D **34**, 3170 (1986).

[10]S. Aoki, to be published.

[11]A better example than the σ model for statistical physicists is the anisotropic spin model.

[12]N. Kawamoto and J. Smit, Nucl. Phys. **B183**, 103 (1981).

[13]S. Aoki, to be published.