Mode Condensation in a Hierarchical System

Deepak Kumar and Kamlesh
Department of Physics, University of Roorkee, Roorkee, Uttar Pradesh 247667, India
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A spherical model of Ising spins, placed on sites that are arranged in a cluster hierarchy, is studied. For the ferromagnetic transition, the critical exponents are found to have nonuniversal values in a certain range of system parameters, while they assume classical values in another range. Correlation functions show a behavior which is very different from the uniform systems. For antiferromagnetic interactions, the system has no transition, which is related to the fact that condensation cannot occur in localized modes.

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It has been realized by many workers that hierarchical organization is an important characteristic of many complex systems.[1-5] The discovery of hierarchical structuring of the pure states of an infinite-ranged spin-glass model by Mezard *et al.*[6] has played a significant role in this realization. Recently, hierarchical models have been used to obtain interesting insights into relaxational dynamics of glassy systems.[1,2,7-11]

In view of the important role that hierarchy might play for complex systems, we study the question of mode condensation in a system with hierarchical architecture. We consider an Ising model in which the spin sites are arranged in a cluster hierarchy, which is exhibited by a Cayley tree as shown in Fig. 1. The spin sites are distributed in clusters of σ sites each, and these clusters are grouped into superclusters of σ clusters each and so on. The spins interact according to the Hamiltonian

$$H = -\frac{1}{2}\sum_{i,j=1}^{N} J_{ij} S_i S_j, \tag{1}$$

where the S_i's are Ising spins, $N = \sigma^n$ is the total number of sites in the system, and n is the number of cluster generations. The interaction matrix $J_{ij} = J_l$, where l is the ultrametric distance[6,7] between sites i and j. The main motivation to study this model comes from the fact that the eigenspectrum of the matrix $\mathbf{J}$ is very different from that of a homogeneous system, in the sense that the localization of its modes ranges from unit ultrametric distance to a distance that covers all the sites in the system.[7,11] At the localized end of the spectrum, the eigenvalues are well spaced, becoming denser and denser as one moves towards more extended modes. By choosing the signs of the J_l's appropriately, one can either make the "extended end" of the spectrum more favorable for condensation or make the "localized end" so.

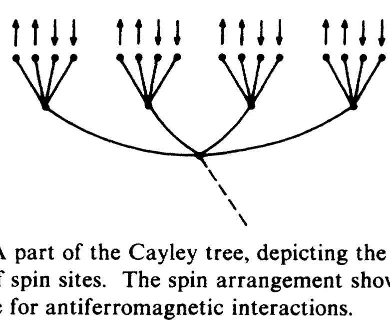

FIG. 1. A part of the Cayley tree, depicting the hierarchical clustering of spin sites. The spin arrangement shows a possible ground state for antiferromagnetic interactions.

Since the branch-point singularity in the mode density at the extended end is quite different from that of uniform systems, one expects a very different type of critical behavior. If, on the other hand, the localized end of the spectrum is more favorable for condensation, another interesting point presents itself. Macroscopic condensation cannot occur in localized modes, as a thermodynamic transition must involve all the degrees of freedom in the system together.[12] For systems possessing both localized and extended modes, Hertz, Fleishman, and Anderson showed, by use of the Hartree-Fock approximation, how condensation in localized modes is averted by a renormalization of staggered susceptibility in these modes.[13] Though the present model does not possess all the interesting features concerning this point, it does offer an alternative mathematical rationale for the Hertz-Fleishman-Anderson result.[14]

To study the statistical mechanics of the above system, we use the spherical model approximation of Berlin and Kac.[15] This approach seems particularly useful in the present context, as it makes the connection between the eigenspectrum of the interaction matrix and the mode condensation and associated critical behavior quite transparent. In the spherical model, the partition function Z can be written as[15]

$$Z = \pi^{N/2} \int_{s_0 - i\infty}^{s_0 + i\infty} \frac{ds}{2\pi} \exp\left[Ns - \frac{1}{2}\sum_{\alpha=1}^{N} \ln\left(s - \frac{\beta\mu_\alpha}{2} \right) \right], \tag{2}$$

where the μ_α's denote the eigenvalues of the matrix $\mathbf{J}$, $\beta = (k_B T)^{-1}$, and s_0 is a real number greater than the largest value of $\beta\mu_\alpha/2$. To proceed further, we briefly recall the main characteristics of the eigenvalue spectrum of $\mathbf{J}$.[7,11] For an n-generation tree, $\mathbf{J}$ has $n+1$ distinct eigenvalues μ_m, which, along with their degeneracies d_m, are given by

$$\mu_m = k\sum_{j=1}^{m-1} J_j\sigma^{j-1} - J_m\sigma^{m-1},$$
$$d_m = k\sigma^{n-m} = Nk/\sigma^m,\quad m=1,\ldots,n; \qquad (3)$$
$$\mu_{n+1} = k\sum_{j=1}^{n} J_j\sigma^{j-1},\quad d_{n+1}=1,$$

where $k=\sigma-1$. The eigenmodes also follow a cluster hierarchy. The first-generation, $m=1$, modes are confined to clusters of σ sites, each such cluster having $\sigma-1$ modes. The next-generation, $m=2$, modes[16] occur in σ clusters of σ sites each, and can be thought of as occurring on supersites obtained by the condensation of σ sites into one supersite as on the second row of the Cayley tree. The $m=3$ modes can be thought of as occurring on single clusters on the third row of the tree, each site of which is obtained by condensation of σ^2 sites. The process is continued till one is left with a single cluster which has σ modes.

We shall now consider a specific ferromagnetic model in which $J_l = Jr^{l-1}$. Then

$$\lambda_m = \mu_m/J = [k-(\sigma-q)q^{m-1}]/(1-q)$$
$$= \lambda_0 - 2bq^{m-1},\quad m=1,\ldots,n; \qquad (4)$$
$$\lambda_{n+1} = \mu_{n+1}/J = \lambda_0(1-q^n),$$

where $q=\sigma r$. The model makes sense only for $q<1$, as otherwise the spectrum of **J** would be unbounded in the $N\to\infty$ limit. For $q<1$, the mode density has a branch-point singularity at λ_0, which is the highest eigenvalue in the limit $n\to\infty$.

Writing $K=\beta J$, $s=Kz$, and $z_0=s_0/K$, we write Eq. (2) as

$$Z = \left(\frac{\pi}{K}\right)^{N/2} K\int_{z_0-i\infty}^{z_0+i\infty}\frac{dz}{2\pi} i\left(z-\tfrac{1}{2}\lambda_{n+1}\right)^{-1/2}\exp\{N[Kz-\tfrac{1}{2}g(z)]\}, \qquad (5)$$

where

$$g(z) = k\sum_{m=1}^{n}\sigma^{-m}\ln\left(z-\tfrac{1}{2}\lambda_0+bq^{m-1}\right). \qquad (6)$$

In the thermodynamic limit, the integral is done by the saddle-point method.[15] The saddle point z_s is given by

$$2K = \frac{k}{\sigma}\sum_{m=0}^{\infty}\sigma^{-m}\left(z_s-\tfrac{1}{2}\lambda_0+bq^m\right)^{-1}. \qquad (7)$$

According to this equation, as K increases z_s moves to the left on the real axis. In order to have a phase transition $g'(z_s)$ must have a finite value when $z_s\to\lambda_0/2$ from above.[15] This requires that $q\sigma>1$, which restricts q to the range $\sigma^{-1}<q<1$. The transition temperature is given by the equation

$$2K_c = g'(\tfrac{1}{2}\lambda_0) = kq/b(\sigma q-1). \qquad (8)$$

To work out the nature of singularity of $g(z)$ at $z=\lambda_0/2$, we put $z_1=(z_s-\lambda_0/2)/b$, and write

$$g'(z) = g'(\tfrac{1}{2}\lambda_0) - \frac{kz_1}{b\sigma}\sum_{m=0}^{\infty}(\sigma q)^{-m}(z_1+q^m)^{-1}. \qquad (9)$$

The second term in Eq. (9) is singular if the above sum diverges for $z_1=0$, which happens if $\sigma q^2<1$. In this case, to leading order in z_1, $g'(z)$ is

$$g'(z) = 2K_c - A_1 z_1^{t_1} + \ldots, \qquad (10)$$

where $t_1=\ln(q\sigma)/|\ln q|$ and $A_1=k/[\sigma b|\ln q|\sin\pi(1-t)]$. Note that $t_1<1$, as a result of the condition $\sigma q^2<1$. On the other hand, if $\sigma q^2>1$ one has to express $g'(z)$ as

$$g'(z) = 2K_c - a_1 z_1 + \frac{kz_1^2}{b\sigma}\sum_{m=0}^{\infty}(\sigma q^2)^{-m}(z_1+q^m)^{-1},$$

where $a_1=kq^2/b(\sigma q^2-1)$. The singularity would now come from the above sum if $\sigma q^3<1$. So in the range $\sigma^{-1/2}<q<\sigma^{-1/3}$,

$$g'(z) = 2K_c - a_1 z_1 + A_2 z_1^{1+t_2} + \ldots. \qquad (11)$$

In general, if $\sigma^{-1/p}<q<\sigma^{-1/(p+1)}$, the leading singular term of $g'(z)$ is of the form $z_1^{p+t_p-1}$, where $t_p=\ln(\sigma q^p)/|\ln q|$. All the t_p's are less than unity. This establishes that in the range $\sigma^{-1}<q<1$, $g'(z)$ has a branch-point singularity and finite value at $\lambda_0/2$. This means that for temperatures $K>K_c$, the saddle point sticks to $\lambda_0/2$, signifying the phase transition.[15] For $q<\sigma^{-1}$, there is no transition.

We can now evaluate the thermodynamic functions and the critical singularities of the response functions. The expressions for specific heat c and the susceptibility χ (per site for each of these quantities) are

$$c/k_B = \begin{cases} \tfrac{1}{2}+2K^2/g''(z_s), & T>T_c, \\ \tfrac{1}{2}, & T<T_c, \end{cases} \qquad (12)$$

$$\chi = (\mu_0^2/J)(z_s-\lambda_0/2),^{-1}\quad T>T_c, \qquad (13)$$

The temperature dependences of these quantities depend upon the value of q. We first discuss the range $\sigma^{-1}<q<\sigma^{-1/2}$, in which range $g'(z)$ is given by Eq. (10). For $K\lesssim K_c$, $z_1\propto(K_c-K)^{1/t_1}$ and

$$g(z_s) = g(\lambda_0/2) + 2bK_c z_1 - A_1' z_1^{1+t_1} + \ldots.$$

From these expressions, the specific-heat exponent α, and the susceptibility exponent γ are found to be $-(t_1^{-1}-1)$ and $1/t_1$, respectively. Thus the exponents depend upon the parameters q and σ, which seem to play a role analogous to dimensionality for this system. The exponent β for the order parameter is found to be $\frac{1}{2}$, which is same as in the uniform case.

We next turn to the calculation of the correlator $C_{ij}=\langle S_iS_j\rangle$ for $K=K_c$. It is easily seen that

$$C_{ij}=K^{-1}\sum_{\nu,m}\frac{\langle \nu m\,|\,i\rangle\langle \nu m\,|\,j\rangle}{z_s-\lambda_m/2}, \qquad (14)$$

where $\langle i\,|\,\nu m\rangle$ and its transpose $\langle \nu m\,|\,i\rangle$ denote the eigenvector of **J**, with m labeling the eigenvalue and ν denoting the degeneracy index. Because of permutation symmetry among sites within clusters and among supersites within superclusters, the correlator $C_{ij}=C_l$, where l is the ultrametric distance between sites i and j. Recalling the characteristics of eigenmodes, we note that only modes of generation l and greater can contribute to $C(l)$. The contribution of lth-generation modes to $C(l)$ works out to be $-\sigma^{-l}(z_s-\lambda_l/2)^{-1}$. The contribution from all the higher-generation modes can be evaluated by our noting that for such modes i and j are condensed to the same supersite, i.e., $\langle i\,|\,\nu m\rangle=\langle j\,|\,\nu r\rangle$ and[11]

$$\textstyle\sum_\nu\langle i\,|\,\nu m\rangle^2=(\sigma-1)/\sigma^m. \qquad (15)$$

Thus

$$KC(l)=-\sigma^{-l}(z_s-\lambda_l/2)^{-1}+k\sum_{m=l+1}^{\infty}\sigma^{-m}(z_s-\lambda_m)^{-1}. \qquad (16)$$

Evaluating the sum by an integral approximation, it is seen that for $l\gg\xi$, where $\xi=|\ln z_1|/|\ln q|$, the sum gives the dominant contribution, and

$$C(l)\propto K^{-1}(K_c-K)^{-1/t_1}q^{(1+t_1)l},\quad l\gg\xi. \qquad (17)$$

If, on the other hand, $l\ll\xi$, the integral contributes to the same order as the first term in Eq. (16) and

$$C(l)=(1/bK)(k/\sigma|\ln q|^{-1}-q)(q\sigma)^{-l}. \qquad (18)$$

Thus, the correlations decay exponentially, the decay being weak in the regime $l\ll\xi$ and rather strong in the regime $l\gg\xi$. In both cases the decay rate is temperature independent, unlike the behavior in uniform systems. The quantity ξ, which determines the regime of strong correlations, behaves like the usual correlation length and diverges like $\ln(K_c-K)$.

A new kind of almost universal behavior is obtained when $q>\sigma^{-1/2}$. This is because in this regime, to leading order, $z_1\propto(K_c-K)$ and

$$g(z_s)=g(\tfrac{1}{2}\lambda_0)+2bK_cz_1-a_1'z_1^2+\dots. \qquad (19)$$

The susceptibility exponent γ is now found to have the classical value of unity. For the specific heat, the leading terms given in Eq. (19) give no singularity at all at the critical point. Keeping the next term then yields for $K\le K_c$ and $q<\sigma^{-1/3}$,

$$\frac{c}{k_B}\simeq\frac{1}{2}-\frac{2K^2}{a_1}\left[1+\frac{A_2''}{a_1}(K_c-K)^{t_2}+\dots\right]. \qquad (20)$$

For $q>\sigma^{-1/3}$, t_2 in Eq. (20) is replaced by unity. Note that the specific heat now has a discontinuity at K_c, since $c=k_B/2$ for $K>K_c$. The exponent of the order parameter remains unchanged. Similarly the expression for the correlator is only slightly modified,

$$C(l)\simeq K^{-1}(K_c-K)^{-1}q^{(1+t_1)l},\quad l\gg\xi,$$

while for $l\ll\xi$, the expression is the same as Eq. (18). Thus, apart from correlations the thermodynamic behavior is completely classical in this regime.

We now consider the antiferromagnetic interaction $J_l=-Jr^{l-1}$. The spectrum is now inverted. The highest eigenvalue is $\mu_1=J$ and its degeneracy is Nk/σ. The ground-state ordering has to occur in one of these modes. The system is frustrated for any value of σ. For even values of σ, the moment of each generation cluster is zero. A typical ground state for $\sigma=4$ is shown in Fig. 1. Because of permutation symmetry any arrangement that permutes spins within clusters will serve the purpose equally well. Let us now consider the possibility of a transition in this situation. The saddle-point equation reads

$$2K=\frac{k}{\sigma}\sum_{m=0}^{\infty}\sigma^{-m}(z_s+\tfrac{1}{2}\lambda_0-bq^m)^{-1}. \qquad (21)$$

Since we now have a series of well-separated poles at the top of the spectrum, the solution of this equation is qualitatively different from the ferromagnetic case, as is clear from Fig. 2. We have a multivalued solution, but the physical considerations select the solution on the rightmost branch. Since one has a solution for any value of K, the system has no phase transition. For the solution on the rightmost branch, it is reasonable to approximate the sum by just one term, i.e., $K=k/2\sigma(z_s-\frac{1}{2})$. This solution may be used to obtain the thermodynamic quantities of interest. The expressions for entropy s and for χ are

$$s/k_B=[1+\ln(\pi k_BT/J)-g(z_s)]/4, \qquad (22)$$

$$\chi=k\mu_0^2/2\sigma k_BT. \qquad (23)$$

Thus in this approximation the susceptibility is seen to have a Curielike behavior right up to zero temperature.

To conclude, we find that the model proposed here exhibits a new variety of critical behavior in the ferromagnetic transition. Since the model as such has no spatial dimension, the parameters σ and q play a role similar to dimension in the sense that the critical indices depend upon them. There is even a phenomenon like upper criti-

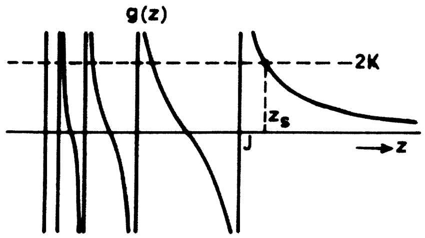

FIG. 2. Graphical solution of Eq. (21).

cal dimension in the sense that critical indices become independent of σ and q. The origin of this peculiar critical behavior lies in the hierarchical organization of the model. For antiferromagnetic interactions, the model shows no phase transition, which is related to the fact that in this case the most favorable modes for condensation are localized.

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[1]R. G. Palmer, D. L. Stein, E. Abrahams, and P. W. Anderson, Phys. Rev. Lett. **53**, 958 (1984).

[2]B. A. Huberman and M. Kerszberg, J. Phys. A **18**, L331 (1985).

[3]H. A. Simon and A. Ando, Econometrica **29**, 111 (1961).

[4]R. H. Austin, K. W. Berson, L. Eisenstein, L. H. Frauenfelder, and I. C. Gunsalus, Biochem. **14**, 5355 (1975).

[5]T. Hogg and B. A. Huberman, Proc. Nat. Acad. Sci., U.S.A. **81**, 6871 (1984).

[6]M. Mezard, G. Parisi, N. Sourlas, G. Toulouse, and M. Virasoro, Phys. Rev. Lett. **52**, 1156 (1984).

[7]A. T. Ogielski and D. L. Stein, Phys. Rev. Lett. **55**, 1634 (1985).

[8]S. Teitel and E. Domany, Phys. Rev. Lett. **55**, 2176 (1985).

[9]D. Kumar and S. R. Shenoy, Solid State Commun. **57**, 927 (1986).

[10]M. Schreckenberg, Z. Phys. B **60**, 4831 (1985).

[11]D. Kumar and S. R. Shenoy, to be published.

[12]P. W. Anderson, Mater. Res. Bull. **5**, 549 (1970).

[13]J. A. Hertz, H. Fleishman, and P. W. Anderson, Phys. Rev. Lett. **43**, 942 (1979).

[14]A. J. Bray and M. A. Moore, J. Phys. C **15**, L765 (1982).

[15]T. H. Berlin and M. Kac, Phys. Rev. **86**, 21 (1952).

[16]The word generation is being used in the backward sense here.