PHYSICAL REVIEW LETTERS

VOLUME 57 | 15 DECEMBER 1986 | NUMBER 24

Finite-Size Scaling and Correlation Lengths for Disordered Systems

J. T. Chayes and L. Chayes
Laboratory of Atomic and Solid State Physics, Cornell University, Ithaca, New York 14853

Daniel S. Fisher
AT&T Bell Laboratories, Murray Hill, New Jersey 07974

and

T. Spencer
The Institute for Advanced Study, Princeton, New Jersey 08540
(Received 10 September 1986)

For a large class of d-dimensional disordered systems, we prove that if an appropriately defined finite-size scaling correlation length diverges at a nontrivial value of the disorder with an exponent ν, then ν must satisfy the bound $\nu \geq 2/d$. Given the assumption that such a correlation length can be defined, the result applies to, e.g., percolation, disordered magnets, and Anderson localization, both with and without interactions. For localization, this puts stringent constraints on scaling theories and interpretation of experiments.

PACS numbers: 05.70.Fh, 02.50.+s, 05.50.+q, 71.30.+h

A number of years ago, Harris[1] argued that if the correlation-length exponent ν of a d-dimensional *uniform* system satisfies $\nu < 2/d$, then the critical behavior of the corresponding disordered system (with random couplings) differs from that of its uniform analog. Thus, for systems where the disorder is irrelevant, ν must satisfy the bound $\nu \geq 2/d$. This condition—which Harris rewrote in terms of the specific-heat exponent α by appeal to the hyperscaling relation $2 - d\nu = \alpha$—is known as the *Harris criterion.* Several authors have discussed the significance of $\nu \geq 2/d$ for other disordered systems, in particular for long-range correlated bond disorder in magnets[2] and for metal-insulator transitions. In the latter case, Mott[3] argued that a minimum metallic conductivity exists only if $\nu \geq 2/d$, but he explicitly did not rule out the possibility that $\nu < 2/d$.

In this Letter, we rigorously establish an inequality of the form $\nu \geq 2/d$ for a large class of disordered systems. We consider any system with independent bond or site disorder, described by a disorder (impurity concentration) parameter λ, $0 \leq \lambda \leq 1$, which undergoes a transition at some value $0 < \lambda_c < 1$. With the assumption that the transition can be characterized (in a sense to be described below) by a change in the scaling behavior of the probability of a system-specific, finite-volume event, this change can be used to define a finite-size scaling correlation length $\xi_f(\lambda)$. We show that if $\xi_f(\lambda)$ diverges as $\lambda \to \lambda_c$ with exponent ν according to $\xi_f(\lambda) \sim |\lambda_c - \lambda|^{-\nu}$, then ν obeys

$$\nu \geq 2/d. \qquad (1)$$

Our proof is quite straightforward and surprisingly general. It follows from a theorem, to be proved below, which bounds the derivative (with respect to impurity concentration) of the probability of *any* finite-volume event in terms of the normal square-root-of-volume fluctuations occurring on the scale at which the event is defined. In particular, unlike Harris's argument, this theorem makes no reference to an analogous uniform system, and thus applies to problems, such as percolation and localization (with or without interactions), which have no uniform analog, as well as to spin-glasses and other disordered magnets.

In order to derive the bound (1) from our theorem, one must identify a system-specific, finite-volume event which is exponentially unlikely at large scales whenever the system is in the "noncooperative" (e.g., nonpercolating, paramagnetic, or localized) phase, but typical on long scales at or beyond the transition point. Since, on general grounds, nothing drastic can occur on the scale of a few lattice spacings, it follows that within the noncooperative phase, a natural length scale emerges beyond which the chosen event becomes unlikely. This scale defines our correlation length, $\xi_f(\lambda)$.

Finite-size scaling correlation lengths have been used frequently in the study of disordered (and uniform) systems, e.g., the Thouless length,[4] which plays a prominent role in the scaling theory of localization,[5] and lengths based on finite-size scaling of "crossing probabilities" in percolation.[6] Nevertheless, it is by no means obvious that any given finite-size scaling correlation length is equivalent—in the sense of critical exponents—to the intrinsic correlation length ξ, defined as, say, the decay rate of correlation functions. Indeed, in systems with first-order transitions or intermediate phases, it is possible that a finite-volume event of the type described above either cannot be defined or will produce a "correlation length" which does not coincide with the intrinsic ξ.[7] However, from general renormalization-group arguments, we expect most critical transitions to be well characterized by the scaling behavior of appropriately chosen finite-volume events. In fact, as discussed below, there are already some systems for which the equivalence of a ξ_f and the intrinsic ξ can be rigorously established.[8]

In the remainder of this Letter, we first consider three systems which are likely to exhibit the behavior described above. Next, we present a general definition of a correlation length in terms of finite-size scaling events. We then prove the theorem concerning normal fluctuations of finite-volume events, and show that it implies (1) for systems of the postulated form. Finally, we discuss the current status of various disordered systems.

We first suggest *finite-size scaling variables* for the illustrative systems. These will be random variables $X_L(\Omega)$, which are functions of the disorder realizations Ω in cubes Λ_L of volume L^d. In particular, we expect that the typical values (e.g., medians) $\tilde{X}_L$ of a natural scaling random variable will obey a finite-size scaling law

$$\tilde{X}_L(\lambda) \sim L^{-y}(L/\xi(\lambda)) \qquad (2)$$

for λ near the critical point and L large. The exponent y and the function f will, of course, depend on the scaling variable X_L. For our purposes, the appropriate variables X_L are those for which $\tilde{X}_L$ decays exponentially for $L \gg \xi$ in the noncooperative phase, but decays no faster than a power law at the critical point.

Examples.—First consider bond percolation[9] at density $0 \le \lambda \le 1$. For any realization Ω, we define $N_L(\Omega)$ to be the number of sites on the left face of Λ_L connected by a path of occupied bonds inside Λ_L to the right. If λ is below the percolation threshold λ_c, and L is large compared with the intrinsic ξ, such paths will be rare and $N_L(\Omega)$ will be zero with probability exponentially (in L) close to 1; above λ_c, $N_L(\Omega)$ should typically be of order L^{d-1}. At λ_c, $N_L(\Omega)$ presumably exceeds some minimal value (e.g., $N_L > 1$) with a nonzero probability which is bounded below uniformly on *all* scales.[9]

Next, consider a random-exchange Ising ferromagnet described by a Hamiltonian with nearest-neighbor couplings J_{xy} independently distributed according to $J_{xy} = J_a$ with probability λ and $J_{xy} = J_b$ with probability $1-\lambda$, with $J_a > J_b \ge 0$. If T_a and T_b are the Curie temperatures of the uniform systems, then for $T_b < T < T_a$, a phase transition occurs at some nontrivial density $\lambda_c(T)$. Consider the dimensionless *interfacial free energy* $\Sigma_L(\Omega)$ of a realization Ω, defined as $1/T$ times the difference in free energy between boxes Λ_L with periodic and antiperiodic boundary conditions in one of the coordinate directions. We expect that in the paramagnetic phase, the probability that Σ_L is larger than any fixed value will be exponentially small (in L), while in the ferromagnet phase, Σ_L will typically be of order L^{d-1}.

Finally, consider the Anderson tight-binding Hamiltonian, $\mathcal{H} = -\Delta + V_x$, where Δ is the lattice Laplacian and the V_x are random potentials with independent and identical distributions.[10] In order to treat this problem on a similar footing to the others considered here, it is convenient (although not necessary[8]) to choose a distribution of potentials characterized by a single impurity-concentration parameter. A simple choice is a convex combination of two symmetric uniform distributions of fixed widths, $W_1 \ll 1$ and $W_2 \gg 1$, which we denote by ρ_{w_1} and ρ_{w_2}, respectively: $\rho_\lambda(V) = \lambda\rho_{w_1}(V) + (1-\lambda)\rho_{w_2}(V)$. In dimensionality $d > 2$, for appropriate choice of w_1 and w_2, it is anticipated that a transition occurs for fixed energy E (near the center of the band) from insulating to metallic behavior at a critical value, $\lambda_c(E)$. We choose for our scaling random variable an appropriate definition of the finite-size *conductance* $g_L(\Omega, E)$ in the box λ_L.[11] One expects that on the localized side of the transition, $\lambda < \lambda_c$, g_L should be exponentially small for large L, while above λ_c, g_L should scale as L^{d-2}; at the transition, it is believed to be typically of order unity (i.e., $e^2/\hbar$) on all scales.[4,5]

Definition of a correlation length.—It seems that the above systems—and presumably others as well—can be described in the vicinity of a critical point by means of the scaling of the probability of the event that a random variable (e.g., N_L, Σ_L, or g_L) is not too small. In particular, for a given choice of variable, X_L, we consider the *finite-size scaling event* $Y_L(a,u) - \{X_L L^u > a\}$, with $a > 0$ fixed and u chosen large enough so that $X_L L^u$ is typically at least of order unity at the transition point, i.e., $u > y$ in (2). Let us suppose that, for some choice of a and u, one can produce a positive constant c so that (A) on a diverging sequence of scales: $\{L_k\}$, $\text{Prob}_{\lambda_c}[Y_{L_k}(a,u)] \geq 2c$, while (B) for $\lambda < \lambda_c$, and for all positive ε, $\text{Prob}_\lambda[X_L L^u > \varepsilon]$ tends to zero exponentially in L. We may then define, for $\lambda < \lambda_c$, a correlation length as the largest scale L at which the probability of $Y_L(a,u)$ exceeds c:

$$\xi_f(\lambda) \equiv \max\{L \mid \text{Prob}_\lambda[Y_L(a,u)] \geq c\}. \tag{3}$$

Notice that $\xi_f(\lambda)$ is defined in terms of the *probability* that a scaling random variable exceeds a certain value, rather than in terms of an *expectation* of the variable. Our definition circumvents difficulties caused by large fluctuations near the critical point, such as those which may occur for the conductance.[12]

The correlation-length bound.—From the above definitions, it is obvious that $\xi_f(\lambda) < \infty$ if $\lambda < \lambda_c$, and that $\xi_f(\lambda_c) = \infty$. If the divergence of ξ_f as $\lambda \to \lambda_c$ is characterized by the critical exponent ν, and if $0 < \lambda_c < 1$, we will show that $\nu \geq 2/d$ in the sense that

$$\limsup_{\lambda \to \lambda_c} \frac{\log \xi_f(\lambda)}{|\log(\lambda - \lambda_c)|} \geq 2/d. \tag{4}$$

To establish (4), we first prove a result concerning the normal fluctuations of finite-volume events[13]:

Theorem.—Consider independently occupied bonds (or sites) of average density λ. If Y is any event depending only on realizations in a finite volume Λ containing $|\Lambda|$ bonds (or sites), then

$$|d\,\text{Pr}_\lambda[Y]/d\lambda| \leq \alpha |\Lambda|^{1/2},$$

where $\text{Pr}_\lambda[Y] \equiv \text{Prob}_\lambda[Y]$ and $\alpha \equiv [\lambda(1-\lambda)]^{-1/2}$.

Proof.—Denoting realizations of bonds (or sites) in Λ by Ω, we have

$$\text{Pr}_\lambda[Y] = \sum \text{Pr}_\lambda[\Omega] E_Y(\Omega),$$

where $E_Y(\Omega)$ is 1 if Y "happens" in the realization Ω and zero otherwise. Since $\text{Pr}_\lambda[\Omega] = \lambda^{n(\Omega)}(1-\lambda)^{|\lambda| - n(\Omega)}$, where $n(\Omega)$ is the number of occupied bonds (or sites) of the realization Ω, it is seen that

$$d\,\text{Pr}_\lambda[\Omega]/d\lambda = \alpha^2[n(\Omega) - \lambda|\Lambda|]\text{Pr}_\lambda[\Omega].$$

Thence, by use of $\lambda|\Lambda| = \langle n \rangle$,

$$|d\,\text{Pr}_\lambda[Y]/d\lambda| \leq \alpha^2 \sum \text{Pr}_\lambda[\Omega]\, |n(\Omega) - \langle n \rangle|.$$

By the Cauchy-Schwarz inequality, the final term is less than $\alpha|\Lambda|^{1/2}$.

Corollary.—For disordered systems with finite-size scaling events obeying conditions (A) and (B) at a nontrivial transition point $0 < \lambda_c < 1$, any consistent choice[14] of $\xi_f(\lambda)$ satisfies (4).

Proof.—At λ_c, we have $\text{Pr}_{\lambda_c}[Y_{L_k}] \geq 2c$, $k = 1, 2, \ldots$, by (A). Using this and integrating the above theorem, we have that for any λ (say) less than λ_c,

$$\text{Pr}_\lambda[Y_{L_k}] \geq 2c - \bar{\alpha}(\lambda_c - \lambda) L_k^{d/2},$$

with $\bar{\alpha}$ (determined by α) nonsingular for $0 < \lambda_c < 1$.[15] Thus, with the choice $\lambda_k \equiv \lambda_c - (c/\bar{\alpha}) L_k^{-d/2}$, it is seen that

$$\xi_f(\lambda_k) \geq L_k = (c/\bar{\alpha})^{2/d}(\lambda_c - \lambda_k)^{-2/d}.$$

Applications and discussion.—We would like to emphasize that the theorem above holds for *any* system with independent bond or site disorder. Thus the applicability of the bound (1) [or (4)] to the conventional ξ reduces to whether one can define finite-size scaling events satisfying (A) and (B), and, if so whether the ξ_f so constructed is at least a lower bound on the intrinsic ξ. It should go without saying that in nonpathological disordered systems with continuous transitions, it would be rather surprising if such scaling events did not exist or did not produce meaningful correlation lengths.

For percolation, it has been proved[6] that a ξ_f defined from a quantity closely related to N_L is equivalent (in the sense of upper and lower bounds which agree up to logarithms) with the intrinsic ξ. Thus, with the exception of the trivial case $d = 1$, the bound (1) holds for the exponent ν.[16]

In the case of random-exchange Ising ferromagnets, we can show[8] equivalence, again in the above sense, of the intrinsic ξ to a ξ_f defined in terms of sums of correlations from the center to the boundary of a finite box. Thus, we have a proof of the bound $\nu \geq 2/d$ for the intrinsic ξ of these systems. Since the Harris criterion is thought to be violated in 4-ε dimensions, this indicates that weak disorder is relevant. The best exponents for the resulting disordered critical point in $d = 3$ are $\nu = 0.70$, found theoretically,[17] and $\nu = 0.73 \pm 0.03$, found experimentally,[18] very close to the bound.

As regards random-field ferromagnets, in $d > 2$ it is now accepted that these systems have a transition for weak disorder,[19] and we expect ν to satisfy (1). Note this does not imply $\alpha < 0$, since here hyperscaling is believed to be violated.

For spin-glasses, the finite-size scaling of a generalized stiffness[20] can be used to define ξ_f. For $d = 3$, numerical estimates[20,21] yield ν considerably larger than $\frac{2}{3}$.

In respect to localization (with interactions), first, it should be noted that the theorem applies to systems with electron-electron interactions and/or spin-flip impurities. Furthermore, it is possible to prove[8] an analogous theorem for derivatives with respect to energy, rather than disorder. Given the appropriate ξ_f, the energy version also implies $\nu \geq 2/d$ for a large class of potential distributions; moreover, it holds in $d = 1$, where the bound is known to saturate (i.e., $\nu = 2$ in $d = 1$).[8] At present, it is not rigorously established that some ξ_f is equivalent to an intrinsic ξ; this is currently under investigation.[8]

Because of the plethora of theoretical and experimental work, our results are probably most useful in the case of localization. Without interactions, extrapolations of the $2+\varepsilon$ expansion for a time-reversal-noninvariant Hamiltonian (e.g., with spin-flip scattering) yield a ν in $d=3$ violating our bound,[22] although higher-order terms could correct this tendency. In the presence of interactions, there has been much confusion arising in part from the question of the relationship between the scaling of the conductivity, $\sigma \sim |\lambda - \lambda_c|^\mu$, and ξ.[23] Without interactions, one expects $\mu = (d-2)\nu$ (so that $g_L \sim 1$ at λ_c), and McMillan[24] proposed that this is also true with interactions. However, later work has suggested[23] this *may* not be the case, although a generally accepted scaling hypothesis (even without calculation of exponents) has not been put forward. Since several experiments apparently show $\mu \approx \frac{1}{2}$ in $d=3$,[25] and dielectric measurements from the localized side have been interpreted as implying $\nu \approx \frac{1}{2}$,[26] our result puts severe constraints on the scaling behavior of localization with interactions and on the analysis of experiments. More direct measurements of the localization length or other characteristic lengths are clearly needed.

We thank E. Abrahams, M. E. Fisher, D. A. Huse, P. A. Lee, J. P. Sethna, and D. J. Thouless for useful discussions. This work was supported in part by National Science Foundation Grant No. DMR-8314625 (for J.T.C.), U.S. Department of Energy Grant No. DE-AC02-83ER13044 (for L.C.), and National Science Foundation Grant No. DMR-8401225 (for T.S.).

[1] A. B. Harris, J. Phys. C 7, 1671 (1974).

[2] A. Weinrib and B. I. Halperin, Phys. Rev. B 27, 413 (1983).

[3] N. F. Mott, Commun. Phys. 1, 203 (1976), and Philos. Mag. B 44, 265 (1981).

[4] D. J. Thouless, Phys. Rev. Lett. 39, 1167 (1977).

[5] E. Abrahams, P. W. Anderson, D. C. Licciardello, and T. V. Ramakrishnan, Phys. Rev. Lett. 42, 673 (1979).

[6] J. T. Chayes, L. Chayes, and J. Fröhlich, Commun. Math. Phys. 100, 399 (1985).

[7] By analogy with the work of V. Privman and M. E. Fisher [J. Stat. Phys. 33, 385 (1983)], one can define a useful finite-size scaling correlation length which diverges at first-order transitions. In the presence of disorder, this should have an exponent ν *equal* to $2/d$ (D. A. Huse, private communication).

[8] J. T. Chayes, L. Chayes, D. S. Fisher, and T. Spencer, to be published.

[9] For precise definitions and results, see, e.g., J. T. Chayes and L. Chayes, in *Critical Phenomena, Random Systems, Gauge Theories,* Proceedings of the Les Houches Summer School, Session XLIII, edited by K. Osterwalder and R. Stora (North-Holland, Amsterdam, 1986).

[10] For precise definitions and results, see, e.g., T. Spencer, in Ref. 9.

[11] Here Ω is a density-λ or -$(1-\lambda)$ sitewise independent realization of "type 1" or "type 2" potentials, distributed according to ρ_{w_1} or ρ_{w_2}, respectively. Thus, Ω corresponds to a set $\{\overline{\Omega}\}$ of realizations $\overline{\Omega}$ of the full disorder distribution, ρ_λ. For each $\overline{\Omega}$, we calculate a (finite) conductance $g_L(\overline{\Omega}, E+i\varepsilon_L)$, where ε_L scales appropriately with L. We then take $g_L(\Omega, E)$ to be, e.g., the median of the $g_L(\overline{\Omega}, E+i\varepsilon_L)$.

[12] P. A. Lee and A. D. Stone, Phys. Rev. Lett. 55, 1622 (1985).

[13] A theorem along these lines was first proved for a restrictive class of events (which does not include the general Y_L) by L. Russo, Z. Wahrsch. Verw. Gebiete 56, 229 (1981).

[14] If X_L were chosen inappropriately, it is possible that the ξ_f defined by X_L would diverge more rapidly than the intrinsic correlation length. Nevertheless, all divergent lengths defined as above diverge at least as rapidly as in (4).

[15] Notice that in $d=1$ (or whenever $\lambda_c = 1,0$), $\bar{\alpha} \sim |\lambda_c - \lambda|^{-1/2}$. This yields the bound $\nu \geq 1/d$, which agrees with the known result $\nu = 1$ for 1D percolation.

[16] A derivation of this result based on different methods has been announced by Durrett and Nguyen; the relevant details can be gleaned from (the union of) R. Durrett and B. Nguyen, Commun. Math. Phys. 99, 253 (1985); B. Nguyen, thesis, University of California, Los Angeles, 1985 (unpublished), and to be published.

[17] K. E. Newman and E. Riedel, Phys. Rev. B 25, 264 (1982).

[18] R. J. Birgeneau, R. A. Cowley, G. Shirane, H. Yoshizawa, D. P. Belanger, A. R. King, and V. Jaccarino, Phys. Rev. B 27, 6747 (1983).

[19] Y. Imry and S. Ma, Phys. Rev. Lett. 35, 1399 (1975); D. S. Fisher, J. Fröhlich, and T. Spencer, J. Stat. Phys. 34, 863 (1984); J. Z. Imbrie, Phys. Rev. Lett. 53, 1747 (1984).

[20] W. L. McMillan, Phys. Rev. B 31, 340 (1985); A. J. Bray and M. A. Moore, Phys. Rev. B 31, 631 (1985); R. J. Caflisch, J. R. Banavar, and M. Cieplak, J. Phys. C 18, L991 (1985).

[21] A. T. Ogielski and I. Morgenstern, Phys. Rev. Lett. 54, 928 (1985); R. N. Bhatt and A. P. Young, Phys. Rev. Lett. 54, 924 (1985).

[22] W. Bernreuther and F. J. Wegner [Phys. Rev. Lett. 57, 1383 (1986)] have done a four-loop calculation and found $\nu = 1/2\varepsilon - \frac{3}{4} + O(\varepsilon)$.

[23] See P. A. Lee and T. V. Ramakrishnan, Rev. Mod. Phys. 57, 287 (1985), and references therein.

[24] W. L. McMillan, Phys. Rev. B 24, 2739 (1981).

[25] M. A. Paalanen, T. F. Rosenbaum, G. A. Thomas, and R. N. Bhatt, Phys. Rev. Lett. 48, 1284 (1982); see also works cited in Ref. 23.

[26] M. Capizzi, G. A. Thomas, F. DeRosa, R. N. Bhatt, and T. M. Rice, Phys. Rev. Lett. 44, 1019 (1980); H. F. Hess, K. DeConde, T. F. Rosenbaum, and G. A. Thomas, Phys. Rev. B 25, 5578 (1982); M. A. Paalanen, T. F. Rosenbaum, G. A. Thomas, and R. N. Bhatt, Phys. Rev. Lett. 51, 1896 (1983).