From Light to Heavy Quarkonia: An Integrated Bethe-Salpeter View

Anuradha Mittal and A. N. Mitra
Department of Physics, University of Delhi, Delhi-7, India
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A modification in the (harmonic) kernel of a QCD-oriented Bethe-Salpeter equation known for good fits to $q\bar{q}$ and qqq spectroscopy gives a comprehensive description of $q\bar{q}$, $Q\bar{q}$, and $Q\bar{Q}$ systems. The new assumptions are (i) proportionality of the "reduced" spring constant $\tilde{\omega}^2$ to α_s ($\tilde{\omega}^2=\omega_0^2\alpha_s$) and (ii) the replacement $r^2 \rightarrow r^2/(1+A_0m_1m_2r^2)^{1/2}-c_0/\omega_0^2$. These provide a more direct QCD motivation for confinement and effect a smooth transition from harmonic ($q\bar{q}$) to linear ($Q\bar{Q}$) confinement.

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The age-old problem of a total understanding of the entire range of hadronic spectra (from the lightest to the heaviest) continues to pose a challenge[1] despite numerous piecemeal successes, especially on the heavy sectors.[2] The natural language of description is still thought to be QCD, but its complex structure does not easily admit of a closed-form solution for confinement (despite great strides in lattice QCD), so that the usual fashion has been to consider *effective* kernels with linear or quadratic behavior. While linear confinement[2] for heavy quarkonia has, by and large, been established, quadratic[3–5] confinement is not ruled out, at least for the light or semilight sectors. The other aspect of the dynamics concerns the role of relativity which, though modest for heavy quarkonia, is thought to be crucial for the light sectors. As a result, the minimum dynamical framework for the implementing of an effective confinement program is provided by the Bethe-Salpeter equation[6,7] in preference to the Schrödinger equation.

A QCD-oriented Bethe-Salpeter approach with quadratic confinement had been proposed some time ago[7] and found to provide a rather good description of light and semilight flavored hadrons ($q\bar{q}$,qqq), not only for their mass spectra[4,5,8] but also other properties (such as pionic and electromagnetic couplings) depending on their (relativistic) wave functions.[9,10] The key ingredients in the $q\bar{q}$ harmonic kernel, which is assumed to have the same spin ($\gamma^{(1)}\gamma^{(2)}$) and color ($\frac{1}{2}\lambda^{(1)}\times\frac{1}{2}\lambda^{(2)}$) dependence as that of the one-gluon-exchange term, are (i) a universal spring constant $\tilde{\omega}^2$ hopefully common to all flavors,[5] and (ii) the quark mass for the flavor sector under study, while the one-gluon-exchange term plays a modest enough role to warrant a mere perturbative inclusion. The model has so far had two major drawbacks: (i) The harmonic-oscillator (h.o.) kernel gives too-large spacings for $Q\bar{Q}$ sectors,[4] and (ii) the experimental $q\bar{q}$ masses require smaller (and mildly flavor-dependent) values of the zero-point energy (ZPE) than the standard h.o. value ($\frac{3}{2}$) for $q\bar{q}$.[4,5,8]

In this Letter we seek to remedy these defects through the following additional assumptions in the h.o. kernel structure:

$$\tilde{\omega}^2=\omega_0^2\alpha_s, \tag{1}$$

$$r^2 \rightarrow r^2-c_0/\omega_0^2, \tag{2}$$

$$r^2 \rightarrow r^2(1+A_0m_1m_2r^2)^{-1/2}, \tag{3}$$

and in the process find an excellent fit to the entire range of meson spectra from $q\bar{q}$ to $Q\bar{Q}$ within a single unified framework.

Assumption (1), which involves the running coupling constant α_s, provides a rather explicit QCD motivation to the kernel and facilitates a further flavor variation in $\tilde{\omega}^2$, so that ω_0^2 is now postulated as a more natural candidate than $\tilde{\omega}^2$, for universal constancy over a wider flavor range. Assumption (2) is designed to account for the ZPE flavor variations in the earlier formalism at the cost of a second constant C_0. Finally the r^2 structure in assumption (3), where m_1,m_2 are the participating quark masses, offers the possibility of an effective *linear* confinement for $Q\bar{Q}$ systems (large m_i), while retaining the harmonic features already tested at the $q\bar{q}$ sectors (small m_i), *provided* the third constant A_0 is small enough.

Before discussing the fits to the data (Tables I–III), we summarize some of the essential features of the calculations whose basic philosophy as well the line of approach have been adequately described in the earlier publications.[4,8] First, the four-dimensional Bethe-Salpeter (BS) equation must be reduced to a three-dimensional form via the null-plane approximation (NPA)[8] in which the relative momentum component q_- plays the same role as does the component q_0 in the instantaneous approximation.[7] Initially, the lack of Lorentz invariance of the h.o. kernel ($\sim r^2$) had stood in the way of a formal NPA covariance of the resulting three-dimensional BS equation in the null-plane variables ($\mathbf{q}_\perp$,q_+), thus making the corresponding wave function unsuitable for applications to processes involving *moving* hadrons. This defect has recently been remedied[11] through a Lorentz-invariant adaptation of the harmonic kernel in momentum space in the fol-

TABLE I. Mass spectra of $c\bar{c}$ and $b\bar{b}$ states (in megaelectronvolts). For input parameters, see text.

Meson	$NJLS$	M (calc.)
$\eta(2980)$	0000	3035
$\psi(3097)$	0101	3092
$\chi(3415)$	1011	3510
$\chi(3510)$	1111	3526
$\chi(3555)$	1211	3556
$\eta(3590?)$	2000	3661
$\psi(3685)$	2101	3696
$\psi(3770)$	2121	3716
$\chi(?)$	3211	4029
$\psi(4030)$	4101	4087
$\psi(4160)$	4121	4101
$\psi(4415)$	6101	4419
$\eta(?)$	0000	9486
$\gamma(9460)$	0101	9498
$\chi(9873)$	1011	9899
$\chi(9894)$	1111	9903
$\chi(9915)$	1211	9912
$\gamma(10\,025)$	2101	10 001
$\chi(10\,254)$	3111	10 269
$\chi(10\,271)$	3211	10 278
$\gamma(10\,355)$	4101	10 321
$\gamma(10\,575)$	6101	10 536

TABLE II. Mass spectra of $q\bar{q}$ ($q=u,d$) and $s\bar{s}$ states (in megaelectronvolts). For input parameters, see text.

Meson	$NJLS$	M (calc.)
$\pi(140)$	0000	163.0
$\rho(775)$	0101	915
$B(1235)$	1110	1182
$f,A_2(1320)$	1211	1352
$\rho'(1600)$	2101	1590
$A(1680)$	2220	1532
$g(1690)$	2321	1682
$h(2030)$	3431	2009
$\rho(2350?)$	4541	2288
$r(2510?)$	5651	2571
$\phi(1020)$	0101	1051
$E(1420)$	1111	1364
$f'(1525)$	1211	1460
$\phi(1680)$	2101	1644
$\phi(1850?)$	2321	1776

lowing sense:

$$\langle p'|K_{\text{h.o.}}|p\rangle = \lim_{m\to 0}\frac{\partial^3}{\partial m^3}\frac{4\pi}{m^2+(p_\mu-p'_\mu)^2},$$

which ensures automatic NPA covariance for the resulting three-dimensional BS equation.[11] However, in the limited context of mass spectra (of immediate interest) it suffices to consider the hadron in its *rest* frame, and hence to take the simpler h.o. representation (r^2).

With use of the techniques of Refs. 4 and 7, and in the notation of Mitra and Mittal,[5] the reduced three-dimensional BS equation in the hadron rest frame, after incorporation of the assumptions (1)–(3), is expressible as

$$[\mathbf{q}^2\gamma^2 - W(\nabla_q^2) - \tfrac{1}{2}\tau_{12}m_{12}\omega_0^2\alpha_s M^{-1}\hat{Q}_q - G(M)]\phi(\mathbf{q}) = 0, \tag{4}$$

$$\gamma^2 = 1 + \left(\frac{m_1^2}{m_2^2}+\frac{m_2^2}{m_1^2}\right)\frac{2\omega_0^2\alpha_s}{m_{12}M} - \frac{2C_0\alpha_s m_{12}\tau_{12}}{M}, \tag{5}$$

$$G(M)/\tau_{12} = \tfrac{1}{4}(M^2-m_{12}^2) - 2m_{12}\omega_0^2\alpha_s M^{-1}(2\mathbf{J}\cdot\mathbf{S}-3) + \tfrac{1}{2}m_{12}MC_0\alpha_s\tau_{12}, \tag{6}$$

$$\tau_{12} = 4m_1m_2m_{12}^{-2}, \quad m_{12} = m_1+m_2, \tag{7}$$

$$W(\nabla_q^2) = \tfrac{1}{2}M\tau_{12}^2\omega_0^2\alpha_s m_{12}\nabla_q^2[1-A_0m_1m_2\nabla_q^2]^{-1/2}, \tag{8}$$

$$\tfrac{1}{4}\hat{Q}_q = \mathbf{q}^2\nabla_q^2 + 2\mathbf{q}\cdot\nabla_q + \tfrac{3}{2}. \tag{9}$$

The solution of Eq. (4) was given in Mitra and Mittal[5] (for the case $A_0=C_0=0$) in the h.o. basis. In particular, the eigenvalues of $\hat{Q}_q$ were determined in terms of SO(2,1) algebra to be given by[5]

$$Q'_N = \langle\tfrac{1}{4}\hat{Q}_q\rangle = -\tfrac{3}{2} - \tfrac{1}{2}(N+\tfrac{3}{2})^2 - \lambda, \tag{10}$$

where $\lambda = -\frac{3}{8}$ ($+\frac{5}{8}$) for even (odd) values, respectively, of the total h.o. quantum number N.

While the constant C_0 term presents no formal difficulty, the treatment of the "potential" term W when $A_0 \neq 0$ may be given either directly in the r representation (which in turn requires a detailed numerical analysis), or preferably in an h.o. basis which has the advantage of exhibiting a greater transparency in its dependence on the total quantum number N with little sacrifice in numerical accuracy. The latter is achieved through the effective replacement

$$(1+A_0m_1m_2r^2)^{1/2} \Rightarrow [1+A_0m_1m_2(N+\tfrac{3}{2})/\beta_N^2]^{1/2} = \sigma_N, \tag{11}$$

TABLE III. Mass spectra of strange, charm, and b-flavor states (in megaelectronvolts). For input parameters, see text.

Meson	NJLS	M (calc.)
$K(496)$	0000	564
$K^*(892)$	0101	983.5
$Q_1(1270)$	1110	1252
$Q_2(1350)$	1111	1312
$K^{**}(1430)$	1211	1412
$L(1770)$	2220	1595
$K^{***}(1780)$	2321	1738
$K^{****}(2060)$	3431	2064
$D(1869)$	0000	2010
$D^*(2010)$	0101	2098
$F(1971)$	0000	2113
$F^*(2140)$	0101	2198
$B(5271)$	0000	5253

where β_N^2 is the inverse range parameter for the Nth state of h.o. excitation which would now be defined nonlinearly through the equation

$$\gamma\beta_N^2(2\sigma_N)^{1/2}=\tau_{12}\omega_0(m_{12}M\alpha_s)^{1/2}. \tag{12}$$

Equation (4) now has the formal solution

$$G(M)+2\tau_{12}m_{12}\omega_0^2\alpha_s Q_N'=2\beta_N^2\gamma^2(N+\tfrac{3}{2}), \tag{13}$$

which does not yet include the (Coulombic) effect of the one-gluon-exchange term. For the running coupling constant we have earlier used the representation[5,12]

$$\alpha_s(M^2)=\frac{12\pi}{(33-2f)}\left(\ln\frac{M^2}{\Lambda^2}\right)^{-1}, \tag{14}$$

$$\Lambda=0.25\text{ GeV},$$

for estimating the (perturbative) effect of the Coulomb term. Its principal features are still retained for the simpler representation obtained by the replacement

$$\alpha_s(M^2)\rightarrow\alpha_s(m_{12}^2), \tag{15}$$

which is what we advocate for its appearance in the *confining term*, in keeping with a natural desire to ensure that the basic dynamics (at source) be independent of the quantity (M) to be determined.

As for the Coulomb term (V_c), while its perturbative inclusion continues to be reasonable for the light-quark sectors, a more accurate treatment is called for in the heavy ($c\bar{c},b\bar{b}$) sectors. This has been achieved through a diagonalization of its matrix elements $\langle n|V_c|n'\rangle$ for different pairs (n,n') of *radial* excitation corresponding to a given set of JLS values. In the present study we have considered a 10×10 diagonalization for S states ($c\bar{c},b\bar{b}$) and 5×5 for all others.

One more item concerns the relativistic correction to the one-gluon exchange, and for this we have considered only the term which is believed to be fairly unambiguous, viz., the spin-dependent ($\sigma_1\cdot\sigma_2$) part of the Fermi-Breit term. It produces the following additive correction to M^2:

$$\delta M^2=\tfrac{64}{9}\pi\sigma_1\cdot\sigma_2\alpha_s(M^2)M^{-1}\tau_{12}^{-1}\xi_n(\beta^2/\pi)^{3/2}, \tag{16}$$

where $\xi_n=1,\frac{2}{3},\frac{8}{15},\ldots$ for the successive radial excitations of the principal ($L=0$) state under study.

The input parameters have been fixed, partly through a prior estimate of the quantities $\tilde{\omega}^2$ ($=\omega_0^2\alpha_s$) and C_0 from light quark spectroscopy,[5] and partly through a search program, starting from the heaviest ($b\bar{b}$) sector which yields the most sensitive determination of A_0. Tables I–III give a comparison of our theoretical values with the data,[13] by use of the following inputs for the basic constants

$$\omega_0^2=0.025\text{ GeV}^2,\quad C_0=0.2958,\quad A_0=0.028\,28 \tag{17}$$

as well as the quark masses (in gigaelectronvolts)

$$m_{ud}=0.27,\quad m_s=0.41,\quad m_c=1.76,\quad m_b=5.21. \tag{18}$$

The most impressive fits are now seen for the $b\bar{b}$ and $c\bar{c}$ sectors which had so far[4,5,8] not been "understood" in this model. The present success has been brought about partly by the flavor variation of α_s, which effects a much needed decrease in the quantity $\tilde{\omega}^2$ ($=\omega_0^2\alpha_s$) from the $q\bar{q}$ to $Q\bar{Q}$ sectors, and partly by the "small" A_0 term which nevertheless dominates through the factor m_1m_2 in the heavy sectors, while playing a negligible role in the *uds* spectroscopy.[14] The last mechanism effectively provides a *linear* confinement for $Q\bar{Q}$ systems in conformity with the usual beliefs,[1,2] without disturbing the quadratic form of confinement found earlier for *uds* spectroscopy.[4,5,8]

For the $q\bar{q}$ sectors, while the quality of fits cannot be judged by the standards of $Q\bar{Q}$ systems, there are nevertheless some important discrepancies for the *ground* state which show a systematic tendency towards overestimation. However, this trend decreases rapidly with L excitation so that the fits are almost perfect for $L\geqslant1$ states (except for A and L). The tendency towards overestimation of the ground-state masses also shows (quite expectedly) a sharp decline with increasing flavor mass, as seen from a comparison of the unequal-flavor sequence K,D,F,B, so that the fit looks almost perfect at the stage of B-meson.

The additive constant C_0/ω_0^2 to the h.o. kernel (r^2) which is designed to simulate the shortfall in the zero-point energy noticed in the earlier formulation,[4,5,8] has played a twofold role: Apart from filling up the zero-point energy shortfall almost perfectly, for all but the $L=0$ states of light $(q\bar{q})$ quarkonia, it has had the effect of narrowing down the earlier variations in the β^2 values (the Gaussian inverse range parameter) for the different $q\bar{q}$ hadrons, a very desirable feature for other hadronic properties which depend on the details of their wave functions. The remaining mass discrepancies for $q\bar{q}$ states of $L=0$ seem to point to some mild flavor variations in C_0, a relatively fine structure effect not pursued in this study.

Finally a word about the pion is in order. Its rather "accurate" mass determination (163 MeV) has been brought about from Eq. (13) corresponding solely to the confining kernel, without, however, our including either the Coulomb or the Fermi-Breit effects. Indeed, the latter would have led to inconsistencies such as $M_\pi^2 < 0$ and/or $\alpha_s < 0$, in view of the unusually small mass (less than Λ) of the pion. Neglect of these terms, only for the pion, had earlier been recommended on very similar grounds.[5] However, this is *not* to claim any authenticity or genuineness about the pion mass result (which is clearly out of step with the main trends), but rather to illustrate the peculiar difficulties associated with this unique particle which has provoked so many authors[15] to give it a special (Goldstone?) status.

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[14]The lack of precision fits to the P states (χ) of $Q\bar{Q}$ warrants the following comments. These fine-structure splittings can be understood quantitatively [S. N. Gupta *et al.*, Phys. Rev. D **31**, 160 (1985)] after the inclusion of various short-range QCD corrections to one-gluon exchange [e.g., E. Eichten *et al.*, Phys. Rev. D **23**, 2726 (1981)]. Being mainly concerned with a unified understanding of both light and heavy sectors, we did not consider these corrections, but since our model has the complete one-gluon exchange plus vector confinement (Ref. 7), these corrections are in principle allowed. The vector confinement gives a $\mathbf{J}\cdot\mathbf{S}$ piece, which is crucial for light-quark systems (Refs. 4,5, and 9), but unimportant for $Q\bar{Q}$—their fine structures come from the above short-range corrections and are insensitive to the long-range confinement, scalar or vector.

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