Quantum Limit for Successive Position Measurements

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We apply a formalism for the description of multitime measurements to the derivation of the quantum limit on the precision with which a pair of successive position measurements can be performed on a free mass under conditions that would be optimal for detecting very weak forces. The result depends on the position resolution of the measuring device as well as on the time interval between the two measurements, and it spans a range of values which include a presently controversial result known as the standard quantum limit; any controversy is thereby resolved.

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Recently there has been a confluence of interest in the quantum-mechanical effects and limitations associated with very small scales and with ultrahigh-precision position measurements. Examples include optical communication, laser interferometry in gravitational-wave detection, and very small-scale solid-state devices.[1] In particular, the quantum-mechanical limitations on the precision of successive position measurements of a free mass are of considerable importance in connection with the detection of gravitational waves. In this connection, a result known as the *standard quantum limit* (SQL) for position measurements [recorded in Eq. (13) below] has been the subject of considerable interest, as well as of some recent controversy.[2] It is the purpose of this Letter to apply a formalism developed for the description of multitime measurements to a derivation of the quantum-mechanical limitation on the precision with which the second of a successive pair of position measurements can be performed with the objective of detecting and measuring very weak forces (or very small accelerations). We shall refer to this result as the *quantum limit,* or the QL [recorded in Eq. (10) below], so as to avoid confusion with the SQL. As will be seen in the following, the crucial feature in the present derivation is a careful definition of the precision in question in reference to the purpose of the underlying experiment and the explicit incorporation of the properties (e.g., finite resolutions) of measuring devices in the description of quantum measurements, a point which has served as a guiding principle in the developments that have led to the present formulation.[3-5] (References 3, 4, and 5 will be referred to as Papers I, II, and III, respectively.) It should be pointed out here that the relevance of the finite resolution of the measuring device was recognized by Caves when he attempted to reestablish the SQL after a serious flaw in the standard argument had been pointed out by Yuen; see the Letters cited in Ref. 2.

The SQL (for position measurements) states that in two successive position measurements of a free mass m, a time T apart, the second measurement cannot be predicted with a dispersion less than $(T/m)^{1/2}$. On the other hand, the above-mentioned Letter by Yuen maintains that the so-called *contractive* states in fact violate the SQL and reduce the dispersion in question below the quoted value. The current (and rather unsettled) state of the issue is summarized in Caves's Letter. The underlying difficulty, as may be seen in the cited works, is a lack in the existing literature on measurement theory of a realistic formulation capable of analyzing problems that arise in actual measurements. We believe the present formalism provides an appropriate means for treating such problems. As we shall see in the following, our analysis shows that the QL in fact spans a range of values depending on the position resolution of the measuring device, and that the lower limit of this range is indeed lower than the SQL value by a factor of $\sqrt{2}$. To provide the physical basis for the following analysis and to arrive at a proper definition of the precision in question, we shall first consider the detection problem that underlies the QL.

Gravitational-wave detection by laser interferometry is based on measurement of the (very weak) force exerted by the passing gravitational wave on (fairly large) masses attached to the end mirrors of a (laser powered) Michelson interferometer by successive measurement of the positions of the mirrors and determination of the resulting acceleration (for details, see the book cited in Ref. 1). In trying to determine the quantum-mechanical limitations imposed on the sensitivity with which forces can be determined by means of successive measurements of the position of a free mass, Braginsky and Vorontsov[2] recognized the reciprocal relation between the uncertainty in the measured value of the force and the time elapsed between the two measurements. This and related uncertainties, commonly referred to as SQL's, originate in the uncertainty in the measured values of the position of the mirrors, hence the focus on the precision of successive position measurements. Clearly then, we must define the precision in question for conditions of optimal sensitivity with respect to the measurement of forces. With this point in mind, we now turn to a description of the measurement in question.

Consider two successive measurements of position at times $-T/2$ and $+T/2$ by means of a device whose *resolution* for position measurements is Δx. This particular measurement, as well as the formalism, the underlying assumptions, and the notation used here, is described in III, where it is shown that the state of the free mass m so measured is given by the density matrix

$$\hat{\rho} = Z^{-1}\exp\{-\sum[\lambda_i^- \hat{\pi}_i^x(-T/2) + \lambda_i^+ \hat{\pi}_i^x(+T/2)]\}. \tag{1}$$

The objects of our attention for the moment are the three variances $(\delta x^{\pm})^2$ and $(\delta X)^2$ defined by

$$(\delta x^{\pm})^2 = \mathrm{Tr}\{\hat{\rho}[\hat{x}(\pm T/2)]^2\} - \{\mathrm{Tr}[\hat{\rho}\hat{x}(\pm T/2)]\}^2,$$
$$(\delta X)^2 = \mathrm{Tr}\{\hat{\rho}[\hat{x}(T/2) - \hat{x}(-T/2)]^2\} - \{\mathrm{Tr}(\hat{\rho}[\hat{x}(T/2) - \hat{x}(-T/2)])\}^2, \tag{2}$$

where, for any operator that does not explicitly depend upon time,

$$\hat{x}(T) = \hat{U}^\dagger(T)\hat{x}\hat{U}(T), \quad \hat{U}(t) = \exp(-it\hat{p}^2/2m),$$

and where the absence of a time argument implies the reference time $t=0$. Note that $(\delta X)^2$ represents the variance in the *displacement* of the mass m. Since this displacement is given by $\hat{X} = \hat{x}(T/2) - \hat{x}(-T/2) = (T/m)\hat{p}$, we find the simple result

$$(\delta X)^2 = (T/m)^2\{\mathrm{Tr}(\hat{\rho}\hat{p}^2) - [\mathrm{Tr}(\hat{\rho}\hat{p})]^2\}. \tag{3}$$

To arrive at the desired limitations on $\delta x^{\pm}$, we find it expedient to consider a unitary transformation implemented by

$$\hat{V} = \exp(im\hat{x}^2/2T)\exp(iT\hat{p}^2/4m).$$

The transformed state $\hat{\rho}_V = \hat{V}\hat{\rho}\hat{V}^\dagger$ is then found to be

$$\hat{\rho}_V = Z^{-1}\exp\{-\textstyle\sum_i[\lambda_i^- \hat{\pi}_i^x + \lambda_i^+ \hat{\pi}_i^q]\}, \tag{4}$$

where we have used $\hat{V}\hat{x}(-T/2)\hat{V}^\dagger = \hat{x}$, and $\hat{V}\hat{x}(+T/2)\hat{V}^\dagger = (T/m)\hat{p}$. The latter quantity, $(T/m)\hat{p}$, is just what we have called $\hat{q}$ in Eq. (4), so that $\hat{\pi}_i^q$ are in fact projection operators for momentum bins of size $\Delta p = (m/T)\Delta x$. Therefore, Eq. (4) describes a *canonical measurement* of state (cf. III) accomplished by means of a position measurement with resolution Δx and a momentum measurement with resolution $\Delta p = (m/T)\Delta x$. Moreover, in the V representation, the variances appear as

$$(\delta x^+)^2 = (T/m)^2\{\mathrm{Tr}(\hat{\rho}_V\hat{p}^2) - [\mathrm{Tr}(\hat{\rho}_V\hat{p})]^2\},$$
$$(\delta x^-)^2 = \{\mathrm{Tr}(\hat{\rho}_V\hat{x}^2) - [\mathrm{Tr}(\hat{\rho}_V\hat{x})]^2\}. \tag{5}$$

In other words, $\delta x^{\pm}$ are respectively equal to $(T/m)\delta p$ and δx *in the new representation.* Our next task is to derive the appropriate quantum-mechanical limitations on $\delta x^{\pm}$ and δX.

The quantity δX, is, by Eq. (3), proportional to the momentum dispersion in the original state $\hat{\rho}$. Precisely this momentum dispersion was considered in III [see Eq. (A1)] where it was shown to be bounded below by $(T/m)^{1/2}$. Hence

$$\delta X \geq (T/m)^{1/2}. \tag{6}$$

Moreover, it was shown in III that the state which minimizes δX is time-reversal invariant, with $\lambda_i^+ = \lambda_i^-$. Since, by Eqs. (1) and (2) above, the two dispersions $\delta x^{\pm}$ are equal whenever this holds, we find that

$$\delta X = (\delta X)_{\min} \text{ implies } \delta x^+ = \delta x^-. \tag{7}$$

As for the quantum-mechanical limitations on $\delta x^{\pm}$ in general, we observe that according to Eqs. (5), $(\delta x^+)(\delta x^-) = (T/m)U_V$, where U_V is the dispersion product $(\delta x)(\delta p)$ for the state $\hat{\rho}_V$. Recall that $\hat{\rho}_V$ represents the result of a canonical measurement by use of devices with position and momentum resolution equal to Δx and $\Delta p = (m/T)\Delta x$, respectively.

Precisely this measurement was considered in II, where we found that U_V has a universal lower bound $U_{\inf}$ which is a function of the dimensionless quantity $k \equiv (2\pi)^{-1}(\Delta x)(\Delta p)$. Moreover, we found that[6]

$$U_{\inf}(k) \simeq \tfrac{1}{2} + (\pi/6)k + O(k^2), \quad k \ll 1,$$
$$U_{\inf}(k) \simeq (\pi/6)k + \cdots, \quad k \gg 1. \tag{8}$$

In particular, in the limit of $k=0$ one has $U_{\inf}(0) = \frac{1}{2}$, which is the standard Heisenberg result. On the other hand, the behavior for $k \gg 1$ is a purely classical result arising from finite resolutions (recall the definition of k given above).

In terms of $U_{\inf}$, we have the general result

$$\delta x^+ \delta x^- \geq (T/m)U_{\inf}((m/2\pi T)(\Delta x)^2). \tag{9}$$

Again, the $k=0$ limit of Eq. (9) is an immediate consequence of the commutator condition $[\hat{x}(T/2), \hat{x}(-T/2)] = -iT/m$.

Having assembled the necessary results, we shall now proceed to the precise definition of the QL for position measurements. As stated in the introductory discussion, we must define the dispersions $\delta x^{\pm}$ for conditions of optimal sensitivity for force measurements. Let us consider, then, following Ref. 2, the case of a mass m subject to a constant force F during a time T. Classically, the mass would undergo a displacement X equal to $(F/2m)T^2$.

As a measure of the uncertainty in the determination of F, one can consider the dispersion $\delta F = (2m/T^2)\delta X$, and regard $(\delta F)_{\min}$ as representing the limit of detectability of forces by successive position measurements.[7] What is crucial here is not the details of how $(\delta F)_{\min}$ is defined but the simple fact that it involves the *displacement* $\hat{X}$ and not the individual positions $\hat{x}(\pm T/2)$, with the immediate consequence that optimal sensitivity in force detection is achieved by those states that minimize δX. Since by Eq. (7) such states possess the property that $\delta x^+ = \delta x^-$, we are led to define the quantum limit for position measurements, l, to be equal to δx^+ for those states $\hat{\rho}$ for which $\delta x^+ = \delta x^-$. Recall that such states are time-reversal invariant (with the origin of time adjusted so that the measurements occur at $t = \pm T/2$). Indeed, one can easily show that the variance in position values as a function of time for such states is given by

$$[\delta x(t)]^2 = l^2 + (\delta X)^2[(t/T)^2 - \tfrac{1}{4}].$$

This formula clearly shows that the time-reversal states are contractive for $-T/2 \le t \le 0$ and expansive for $0 \le t \le T/2$, and that $\langle \hat{x}(-T/2)\hat{p} + \hat{p}\hat{x}(-T/2)\rangle = -(m/T)(\delta X)^2$ is negative definite [cf. Yuen,[2] Eqs. (1) and (2), and remarks subsequent thereto].

We are now in a position to state the results of the above analysis. Using Eq. (9) together with $l = \delta x^+ = \delta x^-$, we arrive at

$$l \ge l_0\{2U_{\inf}((m/2\pi T)(\Delta x)^2)\}^{1/2} \quad \text{(QL)}, \tag{10}$$

where we have defined $l_0 \equiv (T/2m)^{1/2}$. Equation (10) is our statement of the quantum limit for successive position measurements. Using the limiting behavior of $U_{\inf}$ given in Eq. (8), we obtain from Eq. (10)

$$l \ge l_0[1 + \tfrac{1}{24}(\Delta x/l_0)^2], \quad \Delta x/l_0 \ll 1, \tag{11}$$

for high-resolution and/or long-duration measurements, and

$$l \ge (1/\sqrt{12})\Delta x, \quad \Delta x/l_0 \gg 1, \tag{12}$$

for low-resolution and/or short-duration measurements. Note that Eq. (12) states a classical result,[8] as does the second member of Eq. (8), and it merely reflects the fact that the finite bin size Δx induces a minimum in the dispersion δx which cannot be reduced below $(1/\sqrt{12})\Delta x$ (corresponding to a uniform spatial distribution confined to a single bin). Other types of "binning" are readily discussed.

For comparison, we note that the SQL gives

$$l \ge \sqrt{2} l_0 \quad \text{(SQL)}, \tag{13}$$

with a lower limit which is independent of Δx and falls between the absolute minimum l_0 seen in Eq. (11) and the classical result given in Eq. (12). One can see from Eqs. (10)–(12) that the two important scales in the problem are the position resolution Δx and the natural quantum scale of the problem l_0. For example, for sufficiently long measurement times T, l_0 can be made arbitrarily large (for a fixed mass m) so as to render the required resolution for achieving the absolute limit a relatively easy task. Physically, this corresponds to the fact that for such long measurement times, the spread in the spatial distribution is enhanced to such a degree as to make the finite bin size Δx inconsequential.

As pointed out in the introductory remarks, the current discussion on the SQL arose in connection with gravitational-wave detectors using laser interferometry. The most optimistic estimates of Δx for these devices place it at or about l_0, i.e., $\Delta x \gtrsim l_0$. On the basis of Eq. (11), then, one would expect that $l \gtrsim l_0$ for such resolutions. However, it should be remembered that the estimated optimal resolution $\Delta x \gtrsim l_0$ is subject to a number of conditions,[9] among which is a stringent requirement on the measurement time T (which must be matched to the interferometer parameters so as to minimize thermal noise), and also that the presently achievable resolutions actually correspond more closely to the limit given in Eq. (12).

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[1] As examples of recent literature, we mention C. Caves *et al.*, Rev. Mod. Phys. **52**, 341 (1980); J. Tucker and M. J. Feldman, Rev. Mod. Phys. **57**, 1055 (1985); *Quantum Optics, Experimental Gravitation and Measurement Theory,* edited by P. Meystre and M. O. Scully (Plenum, New York, 1983); Y. Srivastava, A. Widom, and M. Friedman, Phys. Rev. Lett. **55**, 2246 (1985).

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[3] H. Partovi, Phys. Rev. Lett. **50**, 1883 (1983).

[4] R. Blankenbecler and H. Partovi, Phys. Rev. Lett. **54**, 373 (1985).

[5] H. Partovi and R. Blankenbecler, preceding Letter [Phys. Rev. Lett. **57**, 2887 (1986).

[6] We have not succeeded in finding a nonperturbative method of calculating $U_{\inf}(k)$ for all k; see Ref. 4.

[7] Note the interesting relation $(\delta F)_{\min} = 2(\delta p)_{\min}/T$, where $(\delta p)_{\min}$ here represents the minimum value of the momentum dispersion in the *original* state $\hat{\rho}$.

[8] To restore Planck's constant to Eqs. (10)–(12), simply replace T by $\hbar T$ in the definition of l_0 and in the argument of $U_{\inf}$.

[9] See the articles on experimental gravitation and laser interferometry in the book cited in Ref. 1.