Time in Quantum Measurements

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Our formalism for describing quantum measurements is generalized to the multitime case and used to derive rigorous time-energy uncertainty relations. For a free particle, we find that $T\delta H \geq \frac{1}{2}$, where δH is the dispersion in energy and T is the measurement duration as given by an external clock of arbitrarily high accuracy. Moreover, any system used as a clock obeys $\delta t \delta H \geq \frac{1}{2}$, where δt is the dispersion in the values of time as measured by the system.

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Recently we presented a formalism for a nonidealized description of quantum measurements. Recognizing that information obtained in a quantum measurement is in general insufficient to determine the state of a system, we developed the statistical mechanics of quantum measurements on the basis of a *maximum uncertainty* principle[1] (Paper II). This principle was in turn inspired by an entropic formulation of uncertainty[2] (Paper I) that adopts the information-theoretic entropy as a measure of uncertainty, and as such it is the expression of the principle of maximum entropy in the context of quantum mechanics. It was shown in II that the maximum-uncertainty principle implies the standard von Neumann expression for ensemble entropy, and thereby provides a unified basis for all of statistical mechanics.

The above developments made no reference to time, as all measurements were assumed to refer to a common instant of time. For example, in the case of position and momentum measurements which we shall refer to as "canonical," we assumed the existence of a device capable of measuring momentum with a given resolution without further analyzing the nature and possible limitations of such a device. In this Letter we extend the formalism so as to describe explicitly both observables of the canonical measurement. This will in turn lead us to consider the occurrence of measurements at more than one time. We then arrive at a multitime generation of our formalism which enables us to analyze a number of long-standing issues regarding the role of time in quantum mechanics in a rigorous way.[3] The main results obtained by means of this analysis are uncertainty relations between momentum and time (Statement A below), and between energy and time (Statements B and C). A related analysis of the quantum limitations on the accuracy of the second of a pair of position measurements of a free particle, recently discussed in the literature in connection with gravitational-wave detection by laser interferometry, is presented in the following Letter.

We start our analysis by examination of the basic, operational meaning of time. Generally speaking, time in a dynamical theory is a *parameter* that marks change; to every (closed) system a Hamiltonian operator $\hat{H}$ may be assigned which determines changes in measured values of the observables of the system by means of the fundamental dynamical equation $d\hat{A} = i[\hat{H},\hat{A}]dt$. Obviously, this parametric description may be rendered coordinate free by comparison of dynamical rates directly, thereby removing all reference to the parameter time. Similarly, time is itself defined and measured self-consistently on the basis of the fundamental dynamical equation. Clearly, time as such has no independent status in dynamics, and any statement regarding time must ultimately be predicated on observed changes in the measured values of the properties of the system. Since information on changes can only come from comparison of data at different times, the necessity of multiple-time measurements becomes evident.[4]

We are thus led to characterize a general quantum mechanical measurement as entailing observables $\hat{A}^{\nu}(t_r^{\nu})$, where ν labels different observables, and r labels the times at which a given measurement is performed (see II for notation). As in I and II, by *measurement* we mean a process that *prepares* a state, and it entails the production of a sufficient number of copies of the system, a fraction of which is subjected to interaction with measuring devices, thereby serving to prepare/measure *reproducibly* the remaining copies. While, on empirical grounds, we believe that there are no fundamental limitations on measurements that cannot be deduced from the above statistical description,[5] and that consequently this formulation is the most general prototype for the description of preparable states, it must be clearly understood that the generality of our results rests on this statistical description of quantum measurements. With this point in mind, we recall from I and II that the measurement of an observable $\hat{A}$ is in general accomplished by means of a measuring device D^A which entails a partitioning of the spectrum of $\hat{A}$ into a number of subsets α_i^A, called bins, with a corresponding decomposition of the Hilbert space into orthogonal subspaces $\mathcal{M}_i^A$ with associated projection operators $\hat{\pi}_i^A$. The measured data are then summarized in a set of probabilities, $\mathcal{P}_i^A$, for finding the outcome of the measurement of $\hat{A}$ to be within the bin α_i^A. We also recall from II that in general the measured data are not sufficient to determine the state of the system (which is specified by a density matrix $\hat{\rho}$); using the maximum uncertainty principle, we proposed that $\hat{\rho}$ be determined by maximization of the ensemble entropy $-\mathrm{tr}(\hat{\rho}\ln\hat{\rho})$, subject to the constraints imposed by the measured data, $\mathcal{P}_i^\nu = \mathrm{tr}(\hat{\pi}_i^\nu \hat{\rho})$.

The novel feature here is the occurrence of nonsimultaneous constraints. However, these may be simply expressed as $\mathcal{P}_{ir}^\nu \equiv \mathcal{P}_i^\nu(t_r^\nu) = \mathrm{tr}[\hat{\pi}_i^\nu(t_r^\nu)\hat{\rho}]$, where $\hat{\pi}_i^\nu(t_r^\nu) = \hat{U}^\dagger(t_r^\nu)\hat{\pi}_i^\nu \hat{U}(t_r^\nu)$. The evolution operator $\hat{U}$ is defined as usual by $(i\partial/\partial t)\hat{U}(t) = \hat{H}\hat{U}(t)$, with $\hat{U}(0)=1$. (In the absence of a time label, the reference time $t=0$ is to be understood.) The density matrix $\hat{\rho}$ is now given by

$$\hat{\rho} = Z^{-1}\exp\left[-\sum_{\nu i r}\lambda_{ir}^\nu \hat{\pi}_i^\nu(t_r^\nu)\right], \qquad (1)$$

following Eq. (3) of II. The partition function Z and the Lagrange multipliers λ are given by $\mathrm{tr}\hat{\rho}=1$ and $\mathcal{P}_{ir}^\nu = (-\partial/\partial\lambda_{ir}^\nu)\ln Z$. The multipliers are constrained to be real by the Hermiticity of $\hat{\rho}$. Note also that since $\hat{\rho}(t) = \hat{U}(t)\hat{\rho}\hat{U}^\dagger(t)$, $\hat{\rho}(t)$ may be obtained from Eq. (1) by our everywhere replacing t_r^ν by $t_r^\nu - t$, as expected from time-translation invariance. It is also worth noting here that t enters the above expressions through the evolution operator $\hat{U}(t)$, and that the set of $\hat{U}(t)$ form an Abelian group which is *parametrized* by t (cf. earlier remarks concerning the meaning of time).

Questions regarding time may now be answered on the basis of Eq. (1). In the following application, we shall apply Eq. (1) to the simple case of a free particle of mass m and Hamiltonian $\hat{H} = \hat{p}^2/2m$ whose state is measured by means of two position measurements at times $t_1 = -T/2$ and $t_2 = T/2$. We shall see in the following Letter that this measurement is in fact equivalent to the canonical (position and momentum) measurement considered in I and II. We assume that the best resolution available for position measurements is Δ (see Ref. 7 in II), corresponding to the bin arrangement $\alpha_i^x = [(i-\frac{1}{2})\Delta, (i+\frac{1}{2})\Delta]$, $i=0,\pm 1,\ldots$. The density matrix that results from this measurement is, following (1),

$$\hat{\rho} = Z^{-1}\exp\{-\textstyle\sum_i[\lambda_i^- \hat{\pi}_i^x(-T/2) + \lambda_i^+ \hat{\pi}_i^x(+T/2)]\}, \qquad (2)$$

where, as before, $\mathcal{P}_i^\mp = (-\partial/\partial\lambda_i^\mp)\ln Z$ are the probabilities obtained from measurements at times $\mp T/2$, respectively. Every physically realizable set of $\mathcal{P}_i^\mp$ (equivalently, every set of real $\lambda_i^\mp$) will determine a state specified by $\hat{\rho}$. Our task below consists in showing that certain uncertainty products involving T, which is the *duration* of the measurement, cannot be reduced below a certain minimum value.

To arrive at uncertainty relations involving T, we first note that the case of $T=0$ actually corresponds to a single position measurement, a case known to fail as a measurement of state (since $\mathrm{tr}\hat{\rho}$ diverges; see II). Therefore, for a measurement to yield a physically acceptable $\hat{\rho}$, we must have $T>0$. With $T>0$ fixed, our first task will be to determine the minimum value of $\delta p = \{\mathrm{tr}(\hat{\rho}\hat{p}^2) - [\mathrm{tr}(\hat{\rho}\hat{p})]^2\}^{1/2}$ as $\lambda_i^\mp$ are varied over all possible (real) values. We shall refer to the states $\hat{\rho}$ resulting from these variations as the set of preparable states (preparable, that is, by the device described above).

Suppose the minimum we are seeking is achieved on $\hat{\rho}_0$; because δp is even under $\hat{\mathcal{T}}$, this minimum will also be achieved on $\hat{\rho}_0^T = \hat{\mathcal{T}}\hat{\rho}_0\hat{\mathcal{T}}^\dagger$, where $\hat{\mathcal{T}}$ is the (antiunitary) time-reversal operator defined by $\hat{\mathcal{T}}\hat{x}\hat{\mathcal{T}}^\dagger = \hat{x}$ and $\hat{\mathcal{T}}\hat{p}\hat{\mathcal{T}}^\dagger = -\hat{p}$. On the other hand, $\hat{\mathcal{T}}\hat{\pi}_i^x(-T/2)\hat{\mathcal{T}}^\dagger = \hat{\pi}_i^x(+T/2)$. The latter, together with (2), shows that $\hat{\rho}^T$ can be obtained from $\hat{\rho}$ by merely the interchange of λ_i^- and λ_i^+, as one should expect on the basis of time-reversal invariance. Hence $\hat{\rho}^T$ is a preparable state if $\hat{\rho}$ is. Moreover, the state $\hat{\rho}_\theta = (\cos^2\theta)\hat{\rho} + (\sin^2\theta)\hat{\rho}^T$ will also be preparable since $\hat{\rho}_\theta$ is experimentally realizable as the given mixture of two preparable states $\hat{\rho}$ and $\hat{\rho}^T$. It then follows that if $\hat{\rho}_0$ minimizes $T\delta p$, so will $\frac{1}{2}(\hat{\rho}_0 + \hat{\rho}_0^T)$. The latter is clearly a preparable and manifestly time-reversal–invariant state. We may therefore assume $\hat{\rho}_0$ to be time-reversal invariant without any loss in generality.

An entirely analogous argument shows that $\hat{\rho}_0$ may be assumed to be parity invariant as well. But then Eq. (2) indicates that for $\hat{\rho}_0$, $\lambda_i^+ = \lambda_i^-$ (time-reversal invariance) and $\lambda_i^\mp = \lambda_{-i}^\mp$ (parity invariance). These two invariances then guarantee that $\hat{\rho}_0$ will also be "Fourier invariant," where the Fourier transformation $\hat{\mathcal{F}}(T)$ is here defined by $\hat{\mathcal{F}}(T)\hat{x}\hat{\mathcal{F}}^\dagger(T) = (T/2m)\hat{p}$ and $\hat{\mathcal{F}}(T)\hat{p}\hat{\mathcal{F}}^\dagger(T) = -(2m/T)\hat{x}$. Note that the kernel $\mathcal{F}(T\,|\,x,x') = (m/\pi T)^{1/2}\exp[(2mi/T)xx']$ realizes the unitary operator $\hat{\mathcal{F}}(T)$ in the $\hat{x}$ representation.

Using the parity and Fourier invariance deduced above, we see that $\mathrm{tr}[\hat{\rho}_0\hat{p}^2] = \mathrm{tr}[\hat{\rho}_0(2m\hat{x}/T)^2]$, so that $(\delta p)^2_{\min} = \frac{1}{2}\mathrm{tr}\{\hat{\rho}_0[\hat{p}^2 + (2m/T)^2\hat{x}^2]\}$. Since the latter is simply the expectation value of a harmonic-oscillator Hamiltonian in the state $\hat{\rho}_0$, we can conclude that $(\delta p)^2_{\min} = m/T$. For the free particle under discussion $\langle\hat{H}\rangle_{\min} = \langle\hat{p}^2/2m\rangle_{\min} = 1/2T$. Thus we have *Statement A:* A free particle of mass m, whose state is measured during a time period T, will have a dispersion in its momentum no less than $(m/T)^{1/2}$ and a mean energy no less than $1/2T$; that is,

$$\delta p \geq (m/T)^{1/2}, \qquad (A1)$$

$$T\langle\hat{H}\rangle \geq \tfrac{1}{2}. \qquad (A2)$$

Note that the lower limits in (A1) and (A2) correspond to a pure state, namely a Gaussian with wave function $\exp(-m\hat{x}^2/T)$. Strictly speaking, such a pure state is inaccessible to actual measurements.

Statement A is a momentum-time uncertainty condition. To find the analogous result for energy and time, a lower bound for $(\delta H)^2 = \langle \hat{H}^2 \rangle - \langle \hat{H} \rangle^2$, the variance in energy, must be determined. Exploiting the Fourier invariance established above, we can write $\langle \hat{p}^4 \rangle = \frac{1}{2}\langle \hat{p}^4 + (2m/T)^4 \hat{x}^4 \rangle$, similarly for $\langle \hat{p}^2 \rangle$, and from these conclude that $(\delta p^2)^2 = \frac{1}{4}(\delta h_+)^2 + \frac{1}{4}(\delta h_-)^2$, where $\hat{h}_\pm = \hat{p}^2 \pm (2m/T)^2 \hat{x}^2$. Now, it can be shown mathematically[6] that $\langle \hat{h}_-^2 \rangle \geq (2m/T)^2$. Since Fourier invariance implies that $\langle \hat{h}_- \rangle = 0$, we see that $(\delta h_-) \geq 2m/T$, and consequently $(\delta p^2) \geq m/T$. This yields *Statement B*: For the measurement described in Statement A, the dispersion in energy will be no less than $1/2T$; or

$$T\delta H \geq \tfrac{1}{2}. \tag{B}$$

Statements A and B above express fundamental limitations in the accuracy of energy (and momentum) measurements arising from the finiteness of the duration of the measurement. A tacit assumption in the above is the existence of (external) clocks of arbitrary accuracy (to measure T). But as pointed out at the outset, time is itself measured by means of changes in nonstationary systems. Therefore any system can in principle serve as a clock, and a moment's thought reveals that there is a reciprocity between the accuracy with which a system can measure time and the dispersion in the measured values of its energy. We now turn to a derivation of this relationship.

Suppose an observable $\hat{A}$ of a system in a state $\hat{\rho}$ is used to measure time (e.g., spin of the cesium atoms in a cesium clock). The system is then a clock and $\hat{A}$ is the *chronometric* property being utilized. Consider a reading of this clock to measure the time of some event. In essence, this corresponds to a measurement of $\hat{A}$ simultaneously with the event, and a mapping[7] of that value onto a corresponding value of time according to the equation of motion $A(t) = \mathrm{tr}[\hat{\rho}(t)\hat{A}]$. Now the measurement of $\hat{A}$ will yield a distribution described by the probability function $\mathcal{P}(A)$, where $\mathcal{P}(A)dA = \mathrm{tr}[\hat{\rho}(t)\hat{\pi}^A(dA)]$, with $\hat{\pi}^A(dA)$ denoting the projection operator onto the spectral interval dA. The operator $\hat{\pi}^A(dA)$ is well defined when $\hat{A}$ is self-adjoint. Clearly, the distribution in the values of A induces a corresponding one in the values of t in the usual way, namely, $\mathcal{P}(t) = [dA(t)/dt]\mathcal{P}[A(t)]$. With $\mathcal{P}(t)$ at hand, we can define $(\delta t)^2 = \int dt\, \mathcal{P}(t)(t-\bar{t})^2$, where $\bar{t} = \int dt\, \mathcal{P}(t)t$. Alternatively, we find

$$(\delta t)^2 = \int dA\, \mathcal{P}(A)[t(A) - \bar{t}^2],$$

where $t(A)$ is the function inverse to $A(t)$.[7] Relating $\mathcal{P}(A)$ back to $\hat{\rho}$, we arrive at a remarkably simple, and intuitively plausible, result:

$$(\delta t)^2 = \mathrm{tr}\{\hat{\rho}(t)[t(\hat{A}) - \bar{t}]^2\}. \tag{3}$$

It should be noted that the variance $(\delta t)^2$ is a joint property of the state of the clock, $\hat{\rho}$, and the chronometric observable $\hat{A}$ (together with the device used to measure $\hat{A}$).

With Eq. (3) at hand, we can use the generalized Heisenberg inequality to conclude that $(\delta t)(\delta H) \geq \frac{1}{2}|\mathrm{tr}\{\hat{\rho}(t)[\hat{H},t(\hat{A})]\}|$; this lower limit will be denoted by $\frac{1}{2}X$. Note that X is simply the magnitude of the rate of change of the operator $t(\hat{A})$ in the state $\hat{\rho}(t)$. Our final task, then, is to minimize X by finding the optimal chronometric observable $\hat{A}_0$. However, if $\hat{A}_0$ is to give rise to an extremum of X, the first-order change in X caused by a change in $\hat{A}$ must vanish. This standard condition requires that $[\hat{H},D(\hat{A})] = 0$, where $D(A) = dt(A)/dA$. The vanishing of the commutator in turn forces $\hat{A}$ to be a function of $\hat{H}$, unless D is the trivial function $D(A) = D_0$, where D_0 is a constant. However, if $\hat{A}$ is a function of $\hat{H}$, $dA(t)/dt$ will vanish, and $\hat{A}$ will not serve to measure time, let alone minimize X (instead, it will maximize it). This leaves $D = D_0$ as the only choice, which in turn implies that $A(t)$ is a linear function of t; with no loss in generality, one can set $A(t) = t$. Thus we have the result that the optimal chronometric observable $\hat{A}_0$ (if it exists) is characterized by the condition $t = \mathrm{tr}[\hat{\rho}(t)\hat{A}_0]$. The corresponding value of X is easily seen to be unity, which therefore implies *Statement C*: The dispersion δt in the values of time measured by means of a system used as a clock cannot be reduced below $(2\delta H)^{-1}$, where δH is the dispersion in the energy of the system; in other words

$$(\delta t)(\delta H) \geq \tfrac{1}{2}. \tag{C}$$

It is worth emphasizing that the above proof does not require the existence of the optimal chronometric observable $\hat{A}_0$. Indeed, the fact that $\hat{H}$ does not in general admit a (well-defined) canonical conjugate[3] shows that $t(\hat{A})$ does not exist in general. Nevertheless, the energy of a quantum system does have an *uncertainty conjugate* which is time as measured by the system itself.

We conclude with a few remarks: (a) The uncertainty relations A, B, and C are consequences of the canonical commutation relations and do not have an independent status. (b) While the lower limits in (A1), (A2), and (C) are greatest lower bounds, the proof we have outlined in Ref. 6 does not establish the same for (B). In any event, actual experiments impose further, often more severe, restrictions arising from finite resolutions, etc., with nontrivial consequences. Papers I and II and the following Letter illustrate examples of these. (c) Bohr's statement of time-energy uncertainty relation essentially corresponds to statement B (cf. discussions relating to Einstein's photon box experiment[3]). (d) A time-energy uncertainty condition first presented by Mandelstam and

Tamm,[3] and subsequently questioned and discussed in the literature, is usually considered to be the only existing one derivable from quantum mechanics.[8] Notwithstanding a bewildering variety of interpretations for it in the literature, the Mandelstam-Tamm result resembles our statement C more closely than it does A or B. (e) Statements A and B, derived for free particles, obviously hold also for a bound particle if T is sufficiently small in comparison with the time scale relevant to the bound state in question.

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[1]R. Blankenbecler and M. H. Partovi, Phys. Rev. Lett. **54**, 373 (1985).

[2]M. H. Partovi, Phys. Rev. Lett. **50**, 1883 (1983).

[3]The very extensive literature on the subject can be traced from M. Jammer, *The Philosophy of Quantum Mechanics* (Wiley, New York, 1977), which also contains a fairly complete discussion of the major works in this area.

[4]Note the restriction to closed systems which excludes, e.g., the case of a particle bound by an external potential.

[5]This proposition clearly bears the influence of Bohr's words that *there can be no limitation on individual measurements that cannot also be obtained from the mathematical formalism and the statistical interpretation,* as quoted by Y. Aharonov and D. Bohm, Phys. Rev. **122**, 1649 (1961).

[6]To simplify the writing, we shall disregard scales and consder $\hat{h}_\pm = \hat{p}^2 \pm \hat{x}^2$ and $[\hat{x},\hat{p}] = i$. Let $\hat{Q} = (\hat{x}\hat{p} + \hat{p}\hat{x})/4$ and $\hat{B} = \exp(-\lambda\hat{Q})\hat{h}_+\exp(\lambda\hat{Q})$. We find that $\hat{B} = \hat{h}_+\cos\lambda - i\hat{h}_-\sin\lambda$. Now for $|\lambda| < \pi/2$, $\hat{B}$ has a point spectrum only which is identical to that of $\hat{h}_+$, i.e., equal to $2n+1$, with $n = 0,1,2,\ldots$. Hence $\langle\psi|\hat{B}\hat{B}^\dagger|\psi\rangle \geq \langle\psi|\psi\rangle$ for any ψ. By considering a sequence $\{\lambda_\nu\}$ that converges to $\pi/2$ from below, one can easily show that $\langle\psi|\hat{h}_-^2|\psi\rangle \geq \langle\psi|\psi\rangle$ for any ψ for which $\langle\psi|\hat{Q}|\psi\rangle$ exists. Restoring scales to $\hat{h}_-$, one recovers the result used in the text.

[7]Of course this correspondence is subject to conditions that ensure the existence of a well-defined, invertible mapping.

[8]Aharonov and Bohm, Ref. 5.