# Fast Magnetic Dynamos in Chaotic Flows

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(Received 19 May 1986)

A generic structure is proposed for rapidly growing magnetic fields in a class of steady, incompressible flows which are everywhere chaotic with positive Liapunov exponent. In such a flow at high magnetic Reynolds number $R_m$, the magnetic field is approximately aligned everywhere with the dilating direction of the flow, and may have extremely complicated spatial structure. In the limit of infinite $R_m$, the leading-order growth rate is bounded below by the positive Liapunov exponent of the flow; thus the field is a fast dynamo.

PACS numbers: 47.65.+a, 05.45.+b, 47.20.Tg

In an electrically conducting fluid in steady flow with velocity field $\mathbf{u}(\mathbf{x})$, the magnetic field satisfies[1]

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}, \quad \nabla \cdot \mathbf{B} = 0, \tag{1}$$

where $\eta$ is the magnetic diffusivity of the fluid. If all lengths and velocities are scaled with respect to the characteristic length $L$ and velocity $U$ of the system as a whole, then the relative strength of the diffusive term is measured by $\varepsilon = R_m^{-1} = \eta/UL$, where $R_m$ is the magnetic Reynolds number of the flow. If the flow is incompressible, the curl in Eq. (1) can be expanded and simplified, yielding the dimensionless equation

$$(\partial_t + \mathbf{u} \cdot \nabla)\mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{u} + \varepsilon \nabla^2 \mathbf{B}. \tag{2}$$

Apart from the diffusive term, (2) is identical to the equation describing the translation, rotation, and stretching of material line elements in the flow. Therefore, it seems plausible that a magnetic field could grow rapidly in time in a chaotic flow with positive Liapunov exponent at high magnetic Reynolds number.

Substituting a field of the form $\mathbf{B}(\mathbf{x},t) = \mathbf{B}_0(\mathbf{x})e^{pt}$ into the dimensionless form of (1) yields (together with appropriate boundary conditions) a vector eigenvalue problem for $\mathbf{B}_0$ and $p$:

$$p\mathbf{B}_0 = \nabla \times (\mathbf{u} \times \mathbf{B}_0) + \varepsilon \nabla^2 \mathbf{B}_0.$$

The taking of the divergence gives $p \nabla \cdot \mathbf{B}_0 = \varepsilon \nabla^2 \nabla \cdot \mathbf{B}_0$, so that for any reasonable boundary conditions $\nabla \cdot \mathbf{B}_0 \equiv 0$ automatically for an unstable $[\mathrm{Re}(p) > 0]$ eigenmode. An important question in astrophysical or geophysical situations is whether, for a given flow, there are eigenmodes with growth rates of order unity (in this scaling) at arbitrarily high magnetic Reynolds numbers. If so, the flow and the corresponding field are collectively called a *fast dynamo.*[2] A particularly simple example of fast-dynamo action in a chaotic flow on a non-Euclidean manifold has been given by Arnol'd *et al.*[3]

From the fluid-mechanical point of view, a good candidate for fast-dynamo action is the *ABC* flow,[4,5] an exact solution of the inviscid Navier-Stokes equations which is periodic in the box $[0,2\pi]^3$, and is chaotic in certain regions.[6,7] Numerical studies have been made of the evolution of a magnetic field with the same periodicity as the flow.[8,9] At moderately high magnetic Reynolds numbers ($R_m = \varepsilon^{-1} \le 200$), dynamo action occurs with a growth rate that is rather insensitive to $R_m$. Soward[10] has recently constructed a remarkable example of fast-dynamo action in a nonchaotic flow that resembles the *ABC* flow with one coefficient (e.g., $C$) zero. In order to obtain strictly fast growth as $\varepsilon \to 0$, however, Soward had to introduce weak singularities into the vorticity field at special points in the flow. Both the *ABC* simulations and Soward's construction have magnetic fields with complicated behavior at the dissipation length scale $\varepsilon^{1/2}$. This agrees with a result of Moffatt and Proctor[11] that the field in a fast dynamo must in general have nontrivial structure at the dissipation scale in order to destroy helicity fluctuations as fast as they are generated.

Theoretical analysis of magnetic fields in flows like the *ABC* flow is very difficult, because the regions of chaos intermingle with regions of regularity in a pattern of fantastic complexity.[7] This paper treats the much simpler problem of the possibility of fast-dynamo action in a class of chaotic flows ($C$ flows[12]) with idealized ergodic and stretching properties. Although this is a highly restricted class, I expect the results to be relevant, both qualitatively and quantitatively, to a large number of other flows which share the properties of chaos and line element stretching. By restricting attention to $C$ flows, I am able to isolate the phenomena associated with chaotic flow and stretching in a much more complete and transparent way than would have been possible with a broader class of flows. On the other hand, by looking at a whole class of flows, rather than one flow in particular, I can distinguish between generic phenomena and effects that might be peculiar to a certain flow.

*Kinematic properties of C flows.*—Given the steady, incompressible flow $\mathbf{u}(\mathbf{x})$, I define the trajectory $\mathbf{X}(\mathbf{a},t)$ as the location at time $t$ of the particle that was at $\mathbf{a}$ at time zero. The relation between $\mathbf{a}$ and $\mathbf{X}(\mathbf{a},t)$ can be thought of as a change of coordinates, with Jacobian (derivative) matrix $M_{ij}(\mathbf{a},t) = \partial X_i(\mathbf{a},t)/\partial a_j$. I shall assume that $\mathbf{u}(\mathbf{x})$ is periodic in space, with bounded fundamental domain $D$ of unit volume; we can therefore think of $\mathbf{u}$ as a flow on $D$ itself, with opposite faces identified in the natural way that yields a three-torus topology. It is also assumed that the velocity field is as bounded, smooth, etc., as necessary.

The properties of a $C$ flow $\mathbf{u}(\mathbf{x})$ that are essential for our analysis are ergodicity and hyperbolicity.[12] The ergodicity property simply says that the long-time average of a given function of space on a typical trajectory equals its integral over $D$. The hyperbolicity property implies that there is a unit-vector field $\hat{\mathbf{e}}_d(\mathbf{x})$, called the dilating vector field, such that

$$C\exp[\lambda t] \le |\mathbf{M}(\mathbf{a},t)\cdot\hat{\mathbf{e}}_d(\mathbf{a})| \le C'\exp[\lambda t] \tag{3}$$

for all positive and negative $t$. $C$, $C'$, and $\lambda$ are positive numbers, and $\lambda$ is known as the positive Liapunov exponent of the flow.

Equation (3) implies that

$$\mathbf{M}(\mathbf{a},T)\cdot\hat{\mathbf{e}}_d(\mathbf{a}) = \exp\left[\int_0^T \Lambda_d(\mathbf{X}(\mathbf{a},t))dt\right]\hat{\mathbf{e}}_d(\mathbf{X}(\mathbf{a},T)), \tag{4}$$

where $\Lambda_d(\mathbf{x}) \equiv \hat{\mathbf{e}}_d\cdot(\hat{\mathbf{e}}_d\cdot\nabla\mathbf{u})$. By use of the ergodicity property, the positive Liapunov exponent can be identified as

$$\lambda = \lim_{T\to\infty}\frac{1}{T}\int_0^T \Lambda_d(\mathbf{X}(\mathbf{a},t))dt = \int_D \Lambda_d(\mathbf{x})\,d^3x. \tag{5}$$

(3) implies that the dilating field satisfies

$$\mathbf{u}\cdot\nabla\hat{\mathbf{e}}_d(\mathbf{x}) = \hat{\mathbf{e}}_d\cdot\nabla\mathbf{u}(\mathbf{x}) - \Lambda_d(\mathbf{x})\hat{\mathbf{e}}_d(\mathbf{x}) \tag{6}$$

and is therefore smooth along trajectories.

Unfortunately, $\hat{\mathbf{e}}_d(\mathbf{x})$ is not smooth in the direction perpendicular to both $\mathbf{u}$ and $\hat{\mathbf{e}}_d$ at $\mathbf{x}$. However, $\hat{\mathbf{e}}_d$ is still a continuous function of position[12]; in fact, there exists a number $E$ related to $C,C'$ such that

$$|\hat{\mathbf{e}}_d(\mathbf{y}) - \hat{\mathbf{e}}_d(\mathbf{x})| \le E\,|\mathbf{x}-\mathbf{y}|^{1/3} \tag{7}$$

for any $\mathbf{x},\mathbf{y}$ sufficiently close together in the domain $D$. In order to use $\hat{\mathbf{e}}_d$ as an ingredient in the construction of a solution to the physical magnetic field problem, it must be smoothed over the smallest length scale likely to appear in the solution, which is the dissipation length $\varepsilon^{1/2}$. Let

$$\mathbf{e}(\mathbf{x}) = \int \varepsilon^{-1/2} s(\varepsilon^{-1/2}|\mathbf{x}-\mathbf{y}|)d^3y\,\hat{\mathbf{e}}_d(\mathbf{y}), \tag{8}$$

where $s(x)$ is a smooth function that is positive for $0 \le x < 1$, zero for $x \ge 1$, and has $4\pi\int_0^1 x^2 s(x)\,dx = 1$. Equation (7) implies $|\mathbf{e}(\mathbf{x}) - \hat{\mathbf{e}}_d(\mathbf{x})| = O(\varepsilon^{1/6})$, and Eqs. (5) and (6) imply

$$\mathbf{u}\cdot\nabla\mathbf{e} = \mathbf{e}\cdot\nabla\mathbf{u} - \Lambda(\mathbf{x})\mathbf{e} + O(\varepsilon^{1/6}), \tag{9}$$

where $\Lambda(\mathbf{x}) = \mathbf{e}\cdot(\mathbf{e}\cdot\nabla\mathbf{u})$, and

$$\int_D \Lambda(\mathbf{x})d^3x = \lambda + \varepsilon^{1/6}\lambda', \tag{10}$$

where $\lambda'(\varepsilon)$ is order 1 as $\varepsilon \to 0$.

*Magnetic field structure.*— Equation (6) shows that if a material line element in a diffusionless fluid is initially aligned with $\hat{\mathbf{e}}_d(\mathbf{x})$ at its starting point, then it remains so aligned as it moves through the flow, and its length increases exponentially as $t\to\infty$. Indeed, for almost all initial orientations, a material line element will tend to align with $\hat{\mathbf{e}}_d$ as $t\to\infty$.[12] This suggests that a good candidate for a fast dynamo in a fluid with dimensionless diffusivity $\varepsilon$ would be a field closely aligned with the smoothed field $\mathbf{e}(\mathbf{x})$ at every point. So let us seek a field of the form

$$\mathbf{B}(\mathbf{x},t) = [\bar\beta(\mathbf{x})\mathbf{e}(\mathbf{x}) + \mathbf{b}(\mathbf{x})]\exp[(\sigma+\sigma')t], \tag{11}$$

with the functions $\mathbf{B},\bar\beta,\mathbf{b}$ periodic in the same domain $D$ as the flow. $\mathbf{b}$ and $\sigma'$ are corrections to the leading-order solution, expected small as $\varepsilon\to 0$. Insertion of (11) into the induction equation (2) yields

$$\mathbf{e}(\sigma+\mathbf{u}\cdot\nabla)\bar\beta + \bar\beta\mathbf{u}\cdot\nabla\mathbf{e} + \sigma'\bar\beta\mathbf{e} + (\sigma+\sigma'+\mathbf{u}\cdot\nabla)\mathbf{b} = \bar\beta\mathbf{e}\cdot\nabla\mathbf{u} + \mathbf{b}\cdot\nabla\mathbf{u} + \varepsilon(\mathbf{e}\nabla^2\bar\beta + 2\nabla\bar\beta\cdot\nabla\mathbf{e} + \bar\beta\nabla^2\mathbf{e}) + \varepsilon\nabla^2\mathbf{b}. \tag{12}$$

For small $\varepsilon$, the diffusive terms are negligible except at scales comparable with the dissipation length $\varepsilon^{1/2}$. At these scales, any significant variations in the field must be due to $\bar\beta(\mathbf{x})$, since $\mathbf{e}(\mathbf{x})$ varies by at most $O(\varepsilon^{1/6})$ over the dissipation length. Consequently, the term $\mathbf{e}\nabla^2\bar\beta$ dominates the other diffusive terms by a factor of $\varepsilon^{-1/6}$. We therefore can expect to get a good description of the field at all length scales by keeping $\varepsilon\mathbf{e}\nabla^2\bar\beta$ as the only diffusive term in the leading-order problem, with the corrections $\mathbf{b}$ and $\sigma'$ expected to be on the order of $\varepsilon^{1/6}$ as $\varepsilon\to 0$. By use of (9), $\mathbf{e}$ can be eliminated from the resulting leading-order equation, leaving a scalar eigenvalue problem for $\bar\beta$ and $\sigma$,

$$(\sigma+\mathbf{u}\cdot\nabla)\bar\beta = \Lambda(\mathbf{x})\bar\beta + \varepsilon\nabla^2\bar\beta. \tag{13}$$

To determine the nature of the most unstable modes of (13), consider a time-dependent field $\beta(\mathbf{x},t)$ satisfying

$$(\partial_t + \mathbf{u}\cdot\nabla)\beta = \Lambda(\mathbf{x})\beta + \varepsilon\nabla^2\beta. \tag{14}$$

Equation (14) has the property that if $\beta$ is initially everywhere positive, then it remains positive everywhere for all future time. For such a solution, we can divide (14) by $\beta$ and integrate over $D$. Referring to (10) for the integral of $\Lambda$ and applying the divergence theorem yields

$$\frac{d}{dt}\int_D \ln\beta(\mathbf{x},t)d^3x = \lambda + \varepsilon\int_D |\nabla\ln\beta|^2 + \varepsilon^{1/6}\lambda'. \tag{15}$$

Integration in time and use of Jensen's inequality[13] then implies that

$$\int_D \beta(\mathbf{x},t)\, d^3x \geq K \exp[(\lambda+\varepsilon^{1/6}\lambda')t], \tag{16}$$

where $K$ depends only on the initial field.

Although (16) places a lower bound on the growth of $\beta$, it is not necessarily a close bound, since small-scale variations in $\beta$ may lead to large values of $\varepsilon|\nabla \ln\beta|^2$. Taking the dot product of (2) with $\mathbf{B}(\mathbf{x},t)$ and averaging over $D$ yields

$$\frac{d}{dt}\int_D |\mathbf{B}(\mathbf{x},t)|^2 d^3x \leq 2S\int_D |\mathbf{B}(\mathbf{x},t)|^2 d^3x, \tag{17}$$

where $S$ is the maximum value of the scalar strain rate $\hat{\mathbf{n}}\cdot(\hat{\mathbf{n}}\cdot\nabla\mathbf{u})$ over all unit vectors $\hat{\mathbf{n}}$ and all points of $D$.

The observations above imply the existence of an eigensolution of (13) whose eigenfunction $\bar{\beta}(\mathbf{x})$ is real and everywhere positive, and whose eigenvalue $\sigma$ is real, with $\lambda+\varepsilon^{1/6}\lambda' \leq \sigma \leq S$. The spatial structure of the corresponding magnetic field is $\mathbf{B}_0(\mathbf{x}) = \bar{\beta}(\mathbf{x})\mathbf{e}(\mathbf{x}) + O(\varepsilon^{1/6})$. To leading order in $\varepsilon^{1/6}$, the growth rate is bounded below by $\lambda$ and above by $S$.

In the simple dynamo of Arnol'd *et al.*,[3] the leading-order field constructed here is actually the exact eigensolution to the full vector eigenproblem, with $\lambda=\sigma=S$.

*Discussion.*—Since numerical[9] and theoretical[10,11] investigations indicate that typical fast-dynamo fields have complicated small-scale structure, it is worthwhile to give a brief intuitive argument for why the fast dynamo constructed in the last section is also likely to possess this kind of structure. Consider a scalar $\phi(\mathbf{x},t)$ that evolves according to the equation

$$(\partial_t + \mathbf{u}\cdot\nabla)\phi = Q(\mathbf{x}) + \varepsilon\nabla^2\phi,$$

$$Q(\mathbf{x}) = \Lambda(\mathbf{x}) + \varepsilon|\nabla \ln\bar{\beta}|^2 - \sigma. \tag{18}$$

$\phi$ can be interpreted as the density field of a passive contaminant which is steadily injected into $D$ by the source $Q$, stirred and mixed by the flow, and eventually dissipated by molecular diffusion. The form of $Q$ has been chosen so that $\phi(\mathbf{x},t)$ approaches $\ln\bar{\beta}(\mathbf{x})$ as $t\to\infty$.

Now, chaotic flows are typically highly efficient at mixing passive contaminants. Regions of concentration variation are continually being stretched and folded by the flow, creating structures at smaller and smaller scales until they are efficiently damped by molecular diffusion.[14] In the large-$t$ limit of (18), this process occurs continuously, with a steady cascade of contaminant moving from the largest scales at which it is injected, to smaller and smaller scales down to the dissipation length. On general grounds, therefore, we can expect the long-time limit of $\phi(\mathbf{x},t)$, and therefore $\bar{\beta}(\mathbf{x})$, to have structure at all scales. This behavior will be reflected in an equally highly convoluted structure for the resulting magnetic field.

A surprising feature of the foregoing analysis is the conclusion that the Liapunov exponent $\lambda$ is a *lower* bound to the growth rate in the limit $\varepsilon\to 0$. The following argument may make this result seem more natural. The Green's function for (14) can be written formally as a "path integral"

$$G(\mathbf{x},t\,|\,\mathbf{a}) = \left\langle \exp\left[\int_0^t \Lambda(\boldsymbol{\xi}(s))ds\right]\right\rangle, \tag{19}$$

in which the angle brackets denote the expectation taken over all Brownian particle paths $\boldsymbol{\xi}(s)$ with $\boldsymbol{\xi}(0)=\mathbf{a}$ and $\boldsymbol{\xi}(t)=\mathbf{x}$. The Brownian paths are material particle trajectories with superposed random displacements to account for the small but finite diffusivity $\varepsilon$. Thus, $G(\mathbf{x},t\,|\,\mathbf{a})$ can be regarded as the expectation of the stochastic differential equation

$$\frac{d}{dt}g(t) = \Lambda(\boldsymbol{\xi}(t))g(t), \quad g(0)=1,$$

over all allowed paths $\boldsymbol{\xi}(t)$.

A crude analog of the $g$ process is the stochastic differential equation

$$\frac{d}{dt}c(t) = l(t)c(t), \quad c(0)=1, \tag{20}$$

where $l(t)$ is a stationary Guassian random function with mean $\langle l(t)\rangle = \bar{l}$ and autocorrelation

$$\langle [l(t_1)-\bar{l}][l(t_2)-\bar{l}]\rangle = L(t_1-t_2).$$

$L(t)$ is assumed to decay rapidly as $t\to\pm\infty$, so that $l(t)$ has a finite correlation time.

(20) has the explicit solution

$$c(t) = \exp\left[\int_0^t l(s)\,ds\right].$$

The Liapunov exponent of the system is defined by

$$\lambda = \lim_{t\to\infty}\frac{1}{t}\ln c(t) = \lim_{t\to\infty}\frac{1}{t}\int_0^t l(s)\,ds, \tag{21}$$

which exists and equals $\bar{l}$ with probability 1. However, the expectation of $c(t)$ is

$$\left\langle \exp\left[\int_0^t l(s)\,ds\right]\right\rangle \approx \exp\left[\bar{l} + \frac{1}{2}\int_{-\infty}^{\infty} L(\tau)\,d\tau\right]t \tag{22}$$

as $t\to\infty$. Thus the existence of finite-time correlations in $l(t)$ results in a growth rate of the expectation of $c(t)$ that is larger than the Liapunov exponent. This demonstration qualitatively explains how the Green's function $G(\mathbf{x},t\,|\,\mathbf{a})$, and therefore the field $\beta(\mathbf{x},t)$, can grow at a rate higher than the Liapunov exponent.

*Conclusion.*—I have considered the possibility of fast-dynamo action in a class of steady, chaotic flows with idealized ergodic and stretching properties. Using a mixture of ideas from magnetohydrodynamics and dynamical systems theory, I have constructed the leading-order approximation to a magnetic field that is rapidly amplified in a given $C$ flow. The construction makes good physical sense and contains the model of Arnol'd *et al.*[3] as a special case. The construction also gives a natural explanation for the origin of the complex small-scale structure seen in the simulations of magnetic fields in the *ABC* flow[9] and Soward's construction,[10] and predicted by Moffatt and Proctor.[11]

The complexity of the dynamos even in the simple situation considered here presents a daunting prospect for the study of dynamos in more complicated and more realistic flows. Nonetheless, it is anticipated that the increased understanding of fast dynamos in these flows will lead to insights into the more important problems of geophysical and astrophysical magnetohydrodynamics.

The ideas of this paper germinated in conversations with Steve Childress. During their development, I benefitted from discussions with S. A. Orszag, V. Yakhot, R. B. Pelz, A. Majda, M. Kruskal, S. Childress, and W. V. R. Malkus. I would like to acknowledge financial support from the National Science Foundation and the Air Force Office of Scientific Research under Contract No. F49620-85-C-0026.

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