# Scaling of the $0^{++}$ Glueball Mass in SU($N$) Hamiltonian Lattice Calculations

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We show that a variational calculation of the ground-state energy of SU($N$) Hamiltonian lattice gauge theories concurrently gives an estimate of the $0^{++}$ glueball mass. Monte Carlo calculations in three spatial dimensions for $N=3$, 4, 5, and 6 indicate that the resulting glueball mass begins to scale for $N\geq 5$.

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Although numerical simulation of lattice gauge theory is currently performed extensively in the Lagrangean[1] form, the alternative Hamiltonian formulation[2] is of direct usefulness in assessing the low-lying spectrum of the theory.[3] In this Letter, we demonstrate that a very simple variational estimate of the $0^{++}$ glueball mass in SU($N$) lattice gauge theory, calculated in the Hamiltonian form, appears to exhibit scaling for $N\geq 5$. The fact that scaling can manifest itself more easily for $N$ large suggests that many ideas of $1/N$ expansion may have practical importance for lattice calculations.

We consider the Kogut-Susskind SU($N$) lattice Hamiltonian[2] in the form

$$H=\frac{g^2}{a}\left\{\sum_l \tfrac{1}{2}E_l^f E_l^f+\frac{2N}{g^4}\sum_p\left[1-\frac{1}{2N}\mathrm{Tr}(U_p+U_p^\dagger)\right]\right\}, \tag{1}$$

where the color-electric field operators are first-order differential operators such that $E_l^a U_l=\delta_{l,l'}T^a U_l$. It has been noted for some time[4,5] that a reasonable trial wave function for the lattice ground state, with a single variational parameter $A$, is

$$\Phi_0=\exp[AN\tfrac{1}{2}\textstyle\sum_p \mathrm{Tr}(U_p+U_p^\dagger)]. \tag{2}$$

This trial function is exact in the strong-coupling limit but is incorrect in the limit of weak coupling. However, since one is only interested in the scaling region rather than the extreme weak-coupling limit, this known defect may not be significant. *A priori,* for nontrivial cases, it is not known whether any physical variables calculated on the basis of this trial function can exhibit scaling. In the past few years, exact Monte Carlo Hamiltonian calculations[6-10] in the case of U(1) and SU(2) have generally confirmed the adequancy of (2) for estimating the lattice ground-state energy. More recently, we have completed an exact Monte Carlo calculation[11] of the SU(3) ground-state energy and found that the variational energies given by $\Phi_0$ were in even better agreement with the exact results than in the case of SU(2). This suggests that $\Phi_0$ may become a better trial function as $N$ increases.

With $\Phi_0$ as the trial function, the variational ground-state energy per plaquette per gluon degree of freedom is given by

$$\frac{\varepsilon_0}{a}=\frac{\langle H\rangle}{N_p(N^2-1)}=\frac{1}{a\xi}\left\{\tfrac{1}{2}AP(A)+\xi^2\frac{2N^2}{N^2-1}[1-P(A)]\right\}, \tag{3}$$

where $\langle O\rangle\equiv\langle\Phi_0|O|\Phi_0\rangle/\langle\Phi_0|\Phi_0\rangle$, $\xi\equiv 1/Ng^2$, and $P(A)$ is the plaquette expectation value defined below. In deriving (3), we have used the fact that

$$\langle -(E_l^f\ln\Phi_0)(E_l^f\ln\Phi_0)\rangle=\langle\tfrac{1}{2}E_l^fE_l^f\ln\Phi_0\rangle, \tag{4}$$

which follows from $\langle\Phi_0|E_l^fE_l^f|\Phi_0\rangle=-\langle E_l^f\Phi_0|E_l^f\Phi_0\rangle$, and that $T^aT^a=(N^2-1)/2N$. As a corollary, (4) also implies that there are two operators,

$$P=(1/N_p)(1/2N)\textstyle\sum_p \mathrm{Tr}(U_p+U_p^\dagger), \tag{5}$$

and

$$\tilde{P}=\frac{AN}{N_p(N^2-1)}\sum_{l,a}\left|E_l^f\frac{1}{2}\sum_p\mathrm{Tr}(U_p+U_p^\dagger)\right|^2, \tag{6}$$

whose expectation values with respect to $\Phi_0^2$ are identical,

$$\langle P\rangle=\langle\tilde{P}\rangle\equiv P(A). \tag{7}$$

The importance of the use of $\tilde{P}$ for estimation of $P(A)$ in a Monte Carlo calculation was first noted elsewhere,[12] and will be further emphasized below. Minimization of $\varepsilon_0$ with respect to $A$ yields

$$\xi^2=\frac{N^2-1}{4N^2}A\left[1+\frac{P(A)}{AP'(A)}\right], \tag{8}$$

 

which determines $A(\xi)$ implicitly as a function of $\xi$.

If $\Phi_0$ is a good approximation to the exact ground state, then the $0^{++}$ glueball mass can be estimated by minimization of

$$M = E_1 - E_0 = \tfrac{1}{2}\langle [F^\dagger,[H,F]]\rangle / \langle F^\dagger F\rangle \tag{9}$$

with respect to $F$, where $F$ is an operator such that $\Phi_1 = F\Phi_0$ is a trial excited state orthogonal to $\Phi_0$. A simple form for $F$, which is exact in the strong-coupling limit, is $F = P - \langle P\rangle$. In this case, (9) is greatly simplified to

$$Ma = g^2\left[\frac{\langle -\frac{1}{2}\sum_l (E_l^f P)(E_l^f P)\rangle}{\langle P^2\rangle - \langle P\rangle^2}\right] \tag{10}$$

$$= \frac{1}{\xi}\left[\frac{N^2-1}{N^2}\frac{P(A)}{AP'(A)}\right] \tag{11}$$

$$= \frac{1}{\xi}\left[4\frac{\xi^2}{A(\xi)} - \frac{N^2-1}{N^2}\right]. \tag{12}$$

In arriving at (11), we have again used (4) and the fact that $P'(A) = 2N^2 N_p[\langle P^2\rangle - \langle P\rangle^2]$. The frontal attack of evaluating (10) directly in a Monte Carlo calculation is disadvantageous because the variance $\langle P^2\rangle - \langle P\rangle^2$ is highly sensitive to correlation effects and *its variance* cannot be reduced independently. In actual calculations, we find it extremely difficult to determine the glueball mass from (10) with any precision. In (12), we have a useful relation which gives the glueball mass directly in terms of the variational parameter $A(\xi)$. Determination of $A(\xi)$ by fitting of the energy minima with parabolas gave reasonable results. However, after exploring various alternatives, we find it simplest to calculate $P(A)$ for a set of values $A_i$, to obtain $P'(A)$ via numerical differentiation, and to determine $\varepsilon_0$ and $M$ as functions of $\xi$ parametrically via (3), (7), and (11). In order to carry out this program, we must be able to evaluate $P(A)$ with fair accuracy. This is fortunately feasible with the help of the alternative operator $\tilde{P}$.

In Monte Carlo calculations, although $\tilde{P}$ is a more complicated operator to evaluate, its variance is much smaller than that of $P$. To evaluate $\tilde{P}$, note that if $U_p = U_1U_2U_3^\dagger U_4^\dagger$, then, since $E^a U^\dagger = -U^\dagger T^a$,

$$\begin{aligned} E_1^a \operatorname{Tr}(U_p + U_p^\dagger) &= -E_4^a \operatorname{Tr}(U_p + U_p^\dagger) \\ &= \operatorname{Tr}[T^a(U_p - U_p^\dagger)], \\ E_2^a \operatorname{Tr}(U_p + U_p^\dagger) &= \operatorname{Tr}[T^a(U_{p1} - U_{p1}^\dagger)], \\ E_3^a \operatorname{Tr}(U_p + U_p^\dagger) &= -\operatorname{Tr}[T^a(U_{p4} - U_{p4}^\dagger)], \end{aligned} \tag{13}$$

where $U_{p1} \equiv U_1^\dagger U_p U_1$ and $U_{p4} \equiv U_4^\dagger U_p U_4$. For $N = 3,4,5$, we evaluate $P(A) = \langle \tilde{P}\rangle$ for eighteen values of $A_i = 0.10(0.02)0.44$ (this notation indicates values from 0.10 to 0.44 with intervals of 0.02). For $N = 6$, the similar range is covered by $A_i = \frac{2}{36}(\frac{1}{36})\frac{19}{36}$. On a $6^3$ lattice, using the standard Metropolis[13] algorithm, we measure $\langle \tilde{P}\rangle$ for a sequence of $A_i$ values by averaging forty block averages of 100 sweeps each. Initially, 400–500 sweeps were used to equilibrate the lattice from a cold start; subsequently, depending on $A_i$, 300 to 500 sweeps were used for reequilibration between successive values of $A_i$. For $N > 3$, generation of the updating SU($N$) matrix is very time consuming; we therefore hit each link only once per sweep. For SU(3), the variance of $\tilde{P}$ is smaller than that of $P$ by more then an order of magnitude near $A \approx 0.10$, and remains a factor of 2 smaller near $A \approx 0.40$. The resulting statistical errors for $P(A)$ are typically less than 0.001 over the designated range of $A$ for all gauge groups.

To perform the numerical differentiation for $P'(A)$, we smoothed the Monte Carlo data with cubic splines and extracted values for $P(A)$ and $P'(A)$ at $A_i = 0.12(0.01)0.42$ and $A_i = \frac{3}{36}(\frac{1}{72})\frac{18}{36}$ for SU(6). The smoothing was very minimal, and was needed only for extraction of intermediate values and derivatives. We have checked that the original data for $P(A)$ are virtually unchanged by the smoothing process and that the values for $P'(A)$ thus obtained for $A_i = 0.12(0.02)0.42$ agree with the more explicit approximation

$$P'(A_i) = [P(A_{i+1}) - P(A_{i-1})]/[A_{i+1} - A_{i-1}]$$

to three significant digits.

Our results for $P(A)$ for SU(3) to SU(6), augmented by interpolated values, are shown in Fig. 1. The statistical errors involved are too small to be shown. The convergence of $P(A)$ toward a large-$N$ limit is plainly visible. Figure 2 gives the corresponding derivatives. This is just the "specific heat" of a three-dimensional Lagrangean lattice gauge theory. The subtle changes in slope are now clearly unmasked. The location of the maximum slope appears to have converged to $A \approx 0.4$

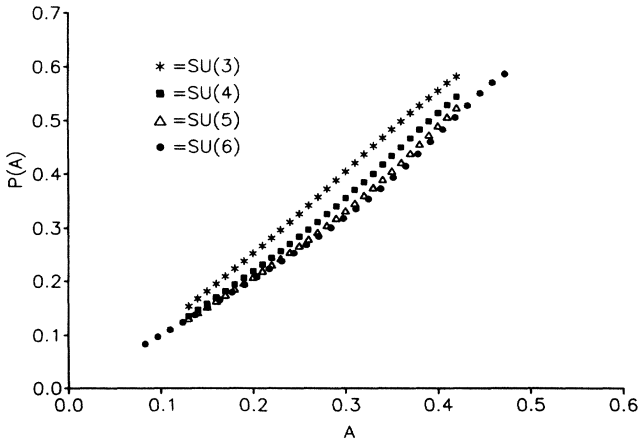


FIG. 1. The plaquette expectation value for SU(3) to SU(6) as a function of the variational parameter $A$ calculated with use of the operator $\tilde{P}$ as defined in Eq. (6). The statistical errors are too small to be shown.

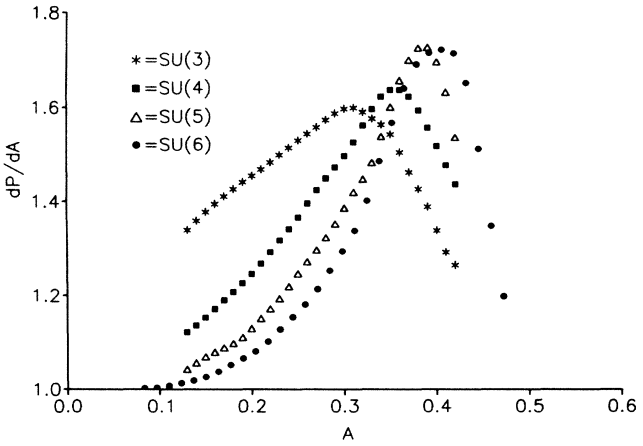


FIG. 2. The derivative of the plaquette expectation value for SU(3) to SU(6) obtained by numerical differentiation of fitted cubic splines. See text for details.

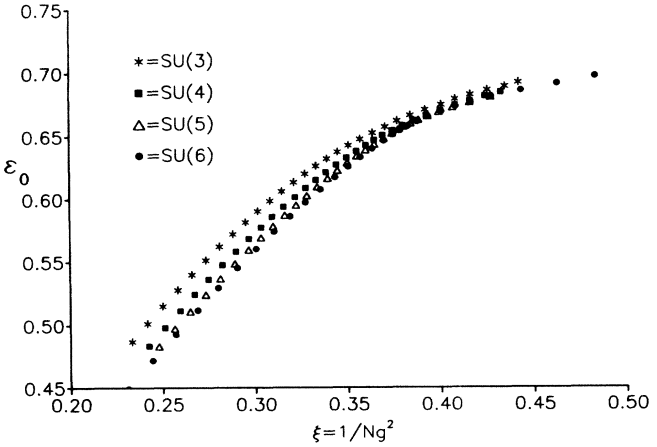


FIG. 3. The variational ground-state energy per plaquette per gluon degree of freedom for SU(3) to SU(6) as a function of coupling constant $\xi=1/Ng^2$.

for large $N$. This is in excellent agreement with the $N=\infty$ limit prediction of a discontinuity in the slope of $P'(A)$ at $A=\pi/8=0.3927$, which is an exact result obtained by Brower, Rossi, and Tan[14] for the simplest possible three-dimensional lattice, that of a single tetrahedron. As shown in Fig. 2, although a discontinuity is not expected for finite $N$, the change in slope near $A=0.4$ can be quite sharp.

Figure 3 shows that the variational energy $\varepsilon_0$ is rather insensitive to $N$. The convergence toward a limiting function at large $N$ is again evident. Since $P(A)$ approaches $1-(N^2-1)/(6AN^2)$ for $A$ large, the variational energy $\varepsilon_0$ approaches $\sqrt{\frac{2}{3}}=0.8165$ in the extreme weak-coupling limit. For comparison, the exact energy in this limit is $\varepsilon_0=0.7959$. Our results for the $0^{++}$ glueball mass, as determined by (8) and (11), are shown in Fig. 4. The solid curves are analytic results obtained by use of known strong-coupling series[15,16] for $P(A)$ up to fifteenth order in $A$. The excellent agreement between analytic results and our $6^3$ lattice data in the strong-coupling regime suggests that the present calculation is not unduly sensitive to finite-size effects. For $N$ large, the dimensionless ratio $Ma/Ng^2$ approaches one in the strong-coupling limit. The straight lines are expected scaling behaviors. In the case of SU(3) and SU(4), no scaling is observed. In view of the crudeness of our operator $F=P-\langle P\rangle$ for estimation of the $0^{++}$ glueball mass, this is not unexpected. [Nevertheless, even in the worse case of SU(3), the glueball-mass minimum as shown is within a factor of 2 of the best Lagrangean estimate.[17]] It is surprising, however, that in the case of SU(5) and SU(6), scaling appears to be obeyed in a narrow stretch of the coupling near $\xi=1/Ng^2=0.38$. We note with interest that in a previous study[18] of string tension in SU($N$) Hamiltonian lattice gauge, scaling behaviors for SU(5) and SU(6) were also inferred near $\xi=0.38$. The drawn scaling lines yield

$$M=(2030\pm 40)\Lambda_L, \tag{14}$$

where $\Lambda_L$ is the conventional Lagrangean lattice scale defined by

$$\frac{\Lambda_L a}{Ng^2}=\xi\left(\frac{48\pi^2}{11}\xi\right)^{51/121}\exp\left(-\frac{24\pi^2}{11}\xi\right). \tag{15}$$

For $N\geq 3$, the ratio of the Hamiltonian lattice scale to the Lagrangean lattice scale is close to unity,[19] and we can therefore safely ignore their distinction.

By demonstrating scaling in a glueball-mass calculation, our work shows that Hamiltonian lattice calculations in general, and variational estimates in particular,

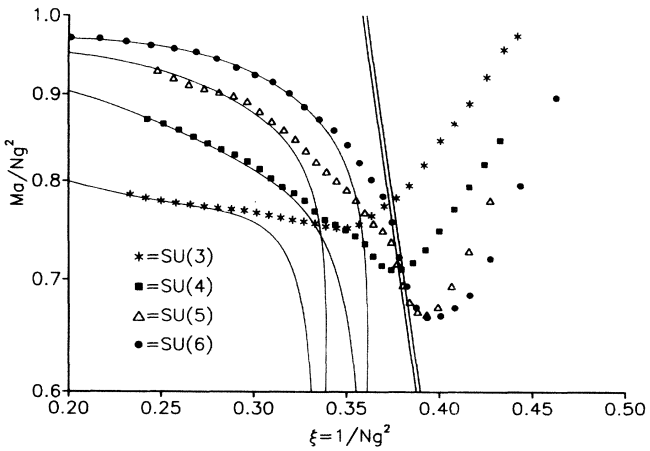


FIG. 4. The variational $0^{++}$ glueball mass for SU(3) to SU(6) as a function of the coupling constant $\xi=1/Ng^2$. The solid curves are strong-coupling-series results and the straight lines are expected scaling behaviors.

are capable of being consistent with the continuum limit. The fact that scaling appears rather easily for $N \geq 5$ suggests that one may profitably approach the physical case of SU(3), *even for Monte Carlo calculations,* from the large-$N$ limit. Of course, it is also very possible that a more refined variational estimate, with improved $\Phi_0$ and $F$, can directly produce a scaling glueball mass for SU(3).

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