Boson Stars: Gravitational Equilibria of Self-Interacting Scalar Fields

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Spherically symmetric gravitational equilibria of self-interacting scalar fields ϕ with interaction potential $V(\phi)=\frac{1}{4}\lambda|\phi|^4$ are determined. Surprisingly, the resulting configuration may differ markedly from the noninteracting case even when $\lambda \ll 1$. Contrary to generally accepted astrophysical folklore, it is found that the maximum masses of such boson stars may be comparable to the Chandrasekhar mass for fermions of mass $m_{\text{fermion}} \sim \lambda^{-1/4} m_{\text{boson}}$.

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Recent speculations in particle physics and cosmology have emphasized the potentially critical role played by evolving scalar fields in the development of the early Universe,[1] and raised the possibility that various exotic bosons comprise the cosmological missing mass.[2,3] In view of this development, a detailed examination of the gravitational equilibria of massive fields in asymptotically flat space-times—"boson stars"—seems appropriate.

Earlier work has considered equilibrium configurations of noninteracting boson fields.[4] Such objects are macroscopic quantum states, that are only prevented from collapsing gravitationally by the Heisenberg uncertainty principle. This requires[5] that a state of characteristic size R has a typical boson momentum $p \sim 1/R$ (here and throughout we set $\hbar \equiv 1 \equiv c$). In a moderately relativistic boson star one has $p \sim m$, where m is the boson mass, so that $R \sim 1/m$. Since hydrostatic equilibrium requires that the total mass $M \sim R/G$ for relativistic bound states, we find $M \sim (Gm)^{-1} = M^2_{\text{Planck}}/m$ for marginally relativistic *boson* stars. This mass is generally much smaller than the Chandrasekhar mass, $M_{\text{Ch}} \sim M^3_{\text{Planck}}/m^2$, characteristic of marginally relativistic *fermion* stars.[6]

In this paper we consider arbitrarily relativistic equilibria of *self-interacting* complex scalar fields. We assume an interaction potential $V(\phi)=\frac{1}{4}\lambda|\phi|^4$ where λ is a dimensionless coupling constant. We demonstrate explicitly that *even if* $\lambda \ll 1$ *the structure of the resulting boson stars may differ radically from the* $\lambda = 0$ *limit.* To see why, first notice that the characteristic energy density inside a $\lambda=0$ configuration is $\rho \sim M^2_{\text{Planck}} m^2$ in the relativistic case. Since the energy density of noninteracting bosons is $\rho \sim m^2|\phi|^2$, we find that $|\phi| \sim M_{\text{Planck}}$ inside relativistic $\lambda=0$ boson stars. Next, consider the effect of increasing λ from zero. The importance of the interaction potential is measured by the ratio $V(\phi)/m^2|\phi|^2$ of interaction energy to kinetic energy. At sufficiently small λ, this ratio is just $\sim \lambda M^2_{\text{Planck}}/m^2$. Thus, self-interactions may only be ignored if

$$\lambda \ll m^2/M^2_{\text{Planck}} = (6.7\times 10^{-39}\ \text{GeV}^{-2})m. \tag{1}$$

Moreover, families of gravitational equilibria may be parametrized by the single dimensionless quantity

$$\Lambda = \lambda M^2_{\text{Planck}}/4\pi m^2. \tag{2}$$

Surprisingly, for $\Lambda \gg 1$ we find that $M \sim \Lambda^{1/2} M^2_{\text{Planck}}/m \sim \lambda^{1/2} M_{\text{Ch}}$, so that *the masses of relativistic boson stars may be comparable to those of their fermion counterparts if* $\lambda \sim 1$.

The starting point for our calculation is the scalar-field Lagrangean

$$\mathcal{L} = -\tfrac{1}{2} g^{\mu\nu}\phi^*_{;\mu}\phi_{;\nu} - \tfrac{1}{2}m^2|\phi|^2 - \tfrac{1}{4}\lambda|\phi|^4, \tag{3}$$

which implies an energy-momentum tensor

$$T^\mu_\nu = \tfrac{1}{2} g^{\mu\sigma}(\phi^*_{;\sigma}\phi_{;\nu} + \phi_{;\sigma}\phi^*_{;\nu}) - \tfrac{1}{2}\delta^\mu_\nu[g^{\lambda\sigma}\phi^*_{;\lambda}\phi_{;\sigma} + m^2|\phi|^2 + \tfrac{1}{2}\lambda|\phi|^4]. \tag{4}$$

We explicitly ignore interactions of ϕ with any other fields. In particular, Eq. (3) assumes negligible coupling to gauge fields, which is a good approximation for gauge coupling constant $e \ll m/M_{\text{Planck}}$. Calculations for larger values of e are currently under way.

We consider spherically symmetric, time-independent solutions of Einstein's field equations

$$G^\mu_\nu = 8\pi G T^\mu_\nu \tag{5}$$

in Schwarzschild coordinates

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2 d\Omega. \tag{6}$$

For such solutions to exist we require that

$$\phi(r,t) = \Phi(r)e^{-i\omega t}, \tag{7}$$

where $\Phi(r)$ is a real function. [We could equally well consider "antiboson stars" with $\phi(r,t)=\Phi(r)e^{i\omega t}$.] For convenience we actually solve the $\binom{t}{t}$ and $\binom{r}{r}$ components of Eq. (5) coupled with the scalar wave equation

$$\phi - m^2\phi - \lambda|\phi|^2\phi = 0, \tag{8}$$

which may be derived from Eq. (5) using Eq. (4) and the Bianchi identities.[6] The resulting equations are

$$\frac{A'}{A^2 x}+\frac{1}{x^2}\left[1-\frac{1}{A}\right]=\left[\frac{\Omega^2}{B}+1\right]\sigma^2+\frac{\Lambda}{2}\sigma^4+\frac{(\sigma')^2}{A}, \tag{9a}$$

$$\frac{B'}{ABx}-\frac{1}{x^2}\left[1-\frac{1}{A}\right]=\left[\frac{\Omega^2}{B}-1\right]\sigma^2-\frac{\Lambda}{2}\sigma^4+\frac{(\sigma')^2}{A}, \tag{9b}$$

$$\sigma''+\left[\frac{2}{x}+\frac{B'}{2B}-\frac{A'}{2A}\right]\sigma'+A\left[\left[\frac{\Omega^2}{B}-1\right]\sigma-\Lambda\sigma^3\right]=0, \tag{9c}$$

where $x=mr$, primes denote d/dx, $\sigma=(4\pi G)^{1/2}\Phi=(4\pi)^{1/2}\Phi/M_{\text{Planck}}$, $\Omega=\omega/m$, and Λ is given by Eq. (2). If we write

$$A(x)=[1-2\mathcal{M}(x)/x]^{-1} \tag{10}$$

we may rewrite Eq. (9a) as

$$\mathcal{M}'(x)=x^2\left[\frac{1}{2}\left[\frac{\Omega^2}{B}+1\right]\sigma^2=\frac{\Lambda}{4}\sigma^4+\frac{(\sigma')^2}{2A}\right]. \tag{9a$'$}$$

Equation (9a′) was substituted for Eq. (9a) in our integrations, with Eq. (10) used to evaluate $A(x)$ in Eqs. (9b) and (9c).

Following Ruffini and Bonazzola[4] we have determined nonsingularly, finite-mass, zero-node solutions to Eqs. (9). It seems reasonable to suppose that these are the lowest-energy bound states since they tend to minimize the contributions of the $(\sigma')^2/2A$ terms in T^μ_ν. Equations (9) are solved subject to the boundary conditions $\mathcal{M}(0)=0$, $\sigma(0)=\sigma_c$, $\sigma'(0)=0$, and $B(\infty)=1$ (i.e., regularity at the origin, asymptotic flatness at infinity). Equations (9a′), (9b), and (9c) constitute an eigenvalue problem for Ω and $B(0)$ subject to the above conditions and the constraint that $\Phi(r)$ have no nodes. Once Ω and $B(0)$ have been determined, the total mass energy $M\equiv\mathcal{M}(\infty)(M^2_{\text{Planck}}/m)$ may be calculated.

Numerical results for $\mathcal{M}(\infty)$ as a function of σ_c are shown in Fig. 1 for several different values of Λ. Our results for $\Lambda=0$ are in excellent agreement with those of Ruffini and Bonazzola.[4]

From Fig. 1, it is apparent that characteristic boson-star masses increase with increasing Λ. To make this trend more quantitative we plot maximum boson-star masses as a function of Λ in Fig. 2. The solid line in Fig. 2 is the relation

$$M_{\max}\approx 0.22\Lambda^{1/2}M^2_{\text{Planck}}/m, \tag{11}$$

which holds asymptotically for $\Lambda\gg 1$ and is derived below.

The key to the derivation may be seen from Fig. 3 where we plot $\sigma(x)$ for $\sigma_c=0.1$ but different values of Λ: $\Lambda=0$ and $\Lambda=300$. This figure illustrates the different structures of models with small and large values of Λ. Whereas $\sigma(x)$ drops smoothly (and ultimately exponentially) to zero in a characteristic length $\sim 1/m$ for small Λ, large-Λ models are characterized by relatively slow decline out to radii $\tau\sim\Lambda^{1/2}/m$ with exponential decay only at larger radii. Most of the mass in the large-Λ configuration is concentrated in the zone of slow decline, which becomes increasingly dominant as Λ increases.

Figure 3 suggests an alternative nondimensionalization

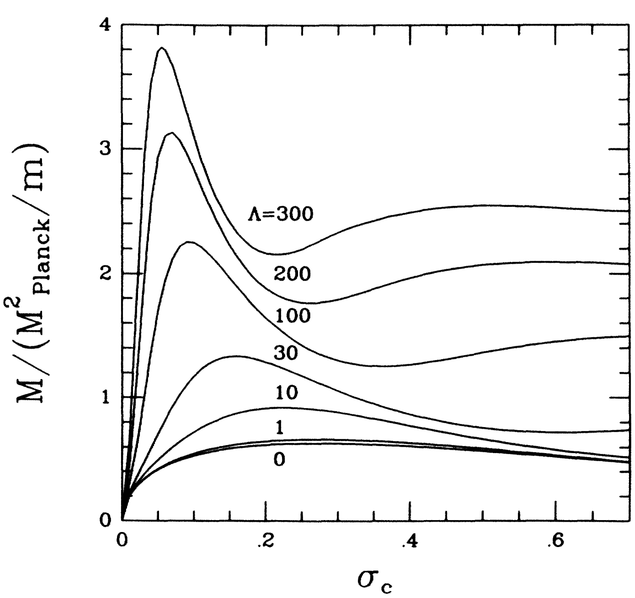

FIG. 1. Boson-star mass M as a function of $\sigma_c=(4\pi G)^{1/2}\Phi_c$ for $\Lambda=0,1$, 10, 30, 100, 200, and 300.

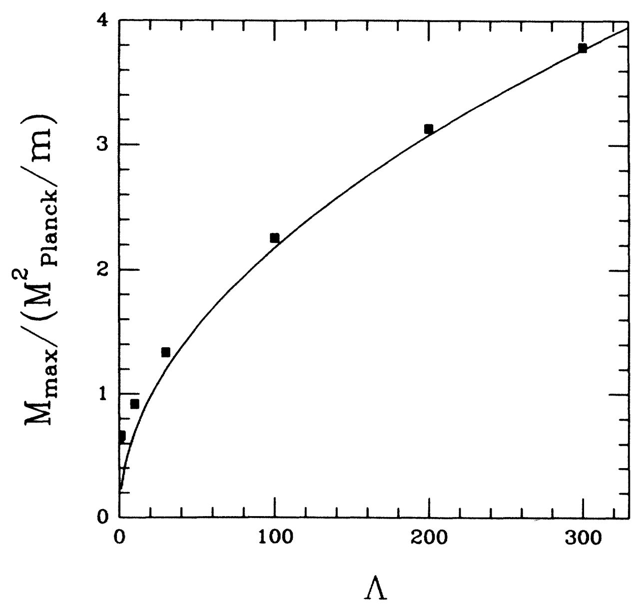

FIG. 2. Maximum boson-star masses as a function of Λ. The solid curve is the asymptotic $\Lambda\rightarrow\infty$ relation Eq. (11).

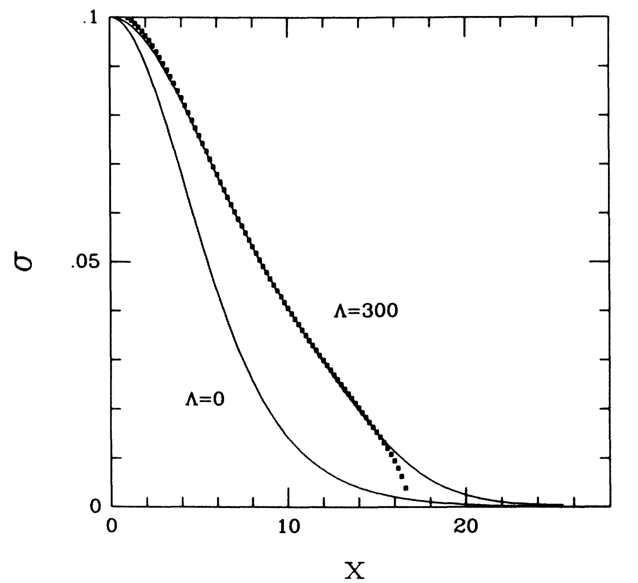

FIG. 3. Scalar field σ as a function of dimensionless radius x for $\sigma_c = 0.1$ but $\Lambda = 0$ and $\Lambda = 300$. The dots are the solution to Eqs. (12)–(15) with $\Omega^2/B(0)$ from the exact $\sigma_c = 0.1$, $\Lambda = 300$ model, scaled to $\Lambda = 300$.

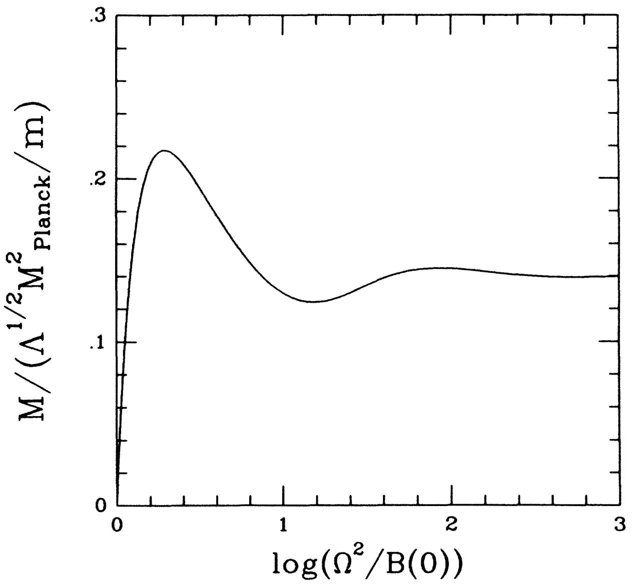

FIG. 4. Boson-star mass vs $\Omega^2/B(0)$ for the limiting equilibria computed from Eqs. (12)–(15).

of Eqs. (9a′), (9b), and (9c) accurate at large Λ: $\sigma_* = \sigma\Lambda^{1/2}$, $x_* = x\Lambda^{-1/2}$, and $\mathcal{M}_* = \mathcal{M}/\Lambda^{1/2}$. Ignoring terms $O(\Lambda^{-1})$, the scalar wave equation may be solved algebraically to yield

$$\sigma_* = (\Omega^2/B - 1)^{1/2}, \tag{12}$$

which may be substituted into the field equations to give, to the same accuracy,

$$\mathcal{M}_*' = 4\pi x_*^2 \rho_* \tag{13}$$

and

$$\frac{B'}{ABx_*} - \frac{1}{x_*^2}\left(1 - \frac{1}{A}\right) = 8\pi p_*, \tag{14}$$

where prime denotes d/dx_* and

$$\rho_* = \tfrac{1}{16}\pi^{-1}(3\Omega^2/B + 1)(\Omega^2/B - 1)^2. \tag{15a}$$

$$p_* = \tfrac{1}{16}\pi^{-1}(\Omega^2/B - 1)^2. \tag{15b}$$

Equations (12)–(15) become essentially exact for $\Lambda \to \infty$. At large Λ these equations may be used to generate approximate solutions. Figure 3 compares $\sigma(x)$ based on Eqs. (12)–(15) with the exact $\sigma(x)$ for $\Lambda = 300$. As expected, the two calculations agree well except at very large radii. Because Λ does not appear explicitly in Eqs. (13)–(15), we can use these limiting equations to determine the rescaled mass $\mathcal{M}_* = M/(\Lambda^{1/2} M_{\mathrm{Planck}}^2/m)$ as a function of the single free parameter $\Omega^2/B(0)$. The results, shown in Fig. 4, imply the peak value $\mathcal{M}_*^{\max} \approx 0.22$ used in Eq. (11).

Because of self-gravity, the ground state of the boson field is *not* a zero-energy state. Moreover, the scalar field, at large Λ, only varies on a relatively large length scale $\Lambda^{1/2} m^{-1} \gg m^{-1}$, so that we can solve the scalar wave equation locally, ignoring derivatives, to get Eq. (12). This results in an effective equation of state, Eqs. (15a) and 15(b), for the boson star, or on elimination of Ω^2/B and restoration of dimensional quantities

$$p = \rho_0 F(\rho/\rho_0), \tag{16}$$

where $\rho_0 = m^4/4\lambda$ and

$$F(\rho/\rho_0) = \tfrac{4}{9}[(1 + \tfrac{3}{4}\rho/\rho_0)^{1/2} - 1]^2. \tag{17}$$

Equation (11) is then equivalent to the statement that $M_{\max} \sim M_{\mathrm{Planck}}^3/\rho_0^{1/2}$ for a fluid star with an equation of state of the form of Eq. (16). It is straightforward to show that the well-known theorems on stability of fluid stars may be applied to the $\Lambda \to \infty$ limiting scalar-field equilibria satisfying Eqs. (12)–(15). In particular, $M_{\max}$ denotes the boundary along the sequence between stable and unstable equilibria as $\Lambda \to \infty$.[7] It seems reasonable to suppose that the same stability criteria may be applied for all (finite) Λ, although we have not proven this statement in general.

Generally accepted astrophysical folklore maintains that boson stars, should they exist, must have negligibly small masses. While detailed studies have corroborated this prejudice for bound states of noninteracting bosons,[4] the situation for self-interacting scalar fields may be very different. In this paper we have shown that for an interaction potential $V(\phi) = \frac{1}{4}\lambda|\phi|^4$ much larger masses will result provided that $\Lambda \equiv \lambda/4\pi G m^2 \gg 1$, an inequality that may be satisfied even at $\lambda \ll 1$ for reasonable scalar-boson masses. When $\Lambda \gg 1$ we have found that the maximum boson-star mass is

$$M_{\max} \approx 0.22\Lambda^{1/2} M_{\text{Planck}}^2/m$$
$$= (0.10\ \text{GeV}^2) M_\odot \lambda^{1/2}/m^2, \qquad (18)$$

which is comparable to the Chandrasekhar mass for fermions of mass $m/\lambda^{1/4}$. It is conceivable that boson stars with masses approaching Eq. (18) could arise in the course of gravitational condensation of bosonic dark matter in the early Universe. Equation (18) would imply masses possibly near, but below, the stellar mass range for scalar neutrinos[2] with $m \sim 1$ GeV, but would appear to require ridiculously large masses, $M_{\max} \sim 10^{27}\lambda^{1/2} M_\odot$, for cosmologically relevant axions[3] (with $m \sim 10^{-5}$ eV) except for exceedingly tiny λ.

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[6]Equation (8) and its complex conjugate may also be derived by considering variations of $S_\phi = \int d^4x\, \mathcal{L}$ with respect to ϕ^* and ϕ, respectively.

[7]See B. K. Harrison, K. S. Thorne, M. Wakano, and J. A. Wheeler, *Gravitation Theory and Gravitational Collapse* (Univ. of Chicago Press, Chicago, 1965), Chaps. 3, 5, and 7, and Appendix B; S. Chandrasekhar, Phys. Rev. Lett. **12**, 114, 437 (1964). Just as for fluid stars, Eq. (14) may be derived by consideration of first-order variations in the total mass, defined in Eq. (13), at fixed boson number, with use of Eqs. (12) and (15). Second-order variations yield the usual stability theory for the analogous fluid stars. Extension of these results to general Λ, for which Eqs. (12) and (15) do not hold, does not appear to be straightforward.