How Large Can a Star Be?

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In a recent article Schoen and Yau develop an inequality which shows that there exists a relationship between the minimum energy density ρ_0 in a star and the size R of the star, of the form $\rho_0 R^2 \leq \pi/6$. This article shows that this inequality is valid for two different measures of the size of the star and conjectures that the inequality can be improved to $\rho_0 R^2 \leq 3\pi/32$.

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A uniformly held view among general relativists is that one cannot put a large amount of matter inside a small volume without causing gravitational collapse. This view is reinforced by the knowledge that for a spherically symmetric distribution of matter, the total mass M cannot exceed $R/2$ (where R is the radius of the distribution, measured in Schwarzschild coordinates, i.e., such that the area of the surface is $4\pi R^2$).

However, a naive calculation for a spherical star of uniform density ρ_0 and radius R would have us believe that the total mass should be the rest mass $(\frac{4}{3}\pi\rho_0 R^3)$ minus the Newtonian binding energy $(\frac{16}{15}\pi^2\rho_0^2 R^5)$,

$$M \approx \tfrac{4}{3}\pi\rho_0 R^3 - \tfrac{16}{15}\pi^2\rho_0^2 R^5. \qquad (1)$$

It is easy to show that this M, for arbitrary choices of ρ_0 and R, never exceeds $R/2$!

Therefore, it is pleasing to discover that an inequality recently announced by Schoen and Yau[1] gives a precise formulation of this idea. Their expression gives a relationship between the energy density of material filling a region and the size of that region. Given a three-dimensional set Ω, which is filled with material whose density ρ is greater than or equal to ρ_0 (some positive constant), they define a measure of the size of Ω, which shall be denoted here as $\mathcal{R}(\Omega)$, and show that

$$\rho_0[\mathcal{R}(\Omega)]^2 \leq \pi/6. \qquad (2)$$

Schoen and Yau[1] directly attack a more geometrical problem. Given a subset Ω of a three-dimensional Riemannian manifold with the three-scalar curvature, ${}^{(3)}R$, greater than or equal to R_0 (a positive constant) on Ω, they show that

$$R_0[\mathcal{R}(\Omega)]^2 \leq 8\pi^2/3. \qquad (3)$$

Condition (3) can be translated into condition (2) if we assume that the Riemannian manifold is part of an initial data set for the Einstein equations. In this case the Hamiltonian constraint[2,3] gives

$${}^{(3)}R - [K_{ij}K^{ij} - (\mathrm{tr}K)^2] = 16\pi\rho \qquad (4)$$

in a system of units where $G = c = 1$. If the slice is maximal, i.e., $\mathrm{tr}K = 0$, we get

$${}^{(3)}R \geq 16\pi\rho. \qquad (5)$$

On substituting (5) into (3) we get (2).

A key part of the Schoen and Yau analysis is their definition of $\mathcal{R}(\Omega)$. It is expressed in terms of the "largest" torus that can be imbedded in Ω. Let Γ be a simple closed curve in Ω. Choose a constant p such that the set of points within a distance p of Γ is contained within Ω and forms a proper torus, i.e., has a hole through the middle. p is a measure of the size of Ω and $\mathcal{R}(\Omega)$ is defined as the largest value of p we can find by considering all curves Γ.

On one level, the idea of the largest imbedded torus is useful in that it conveys the notion that Ω must be large in all three directions. On the other hand, it is hard to evaluate in practice.

In proving Theorem 1 of Ref. 1, Schoen and Yau consider a minimal-area two-surface Σ imbedded in a three-manifold N. Consider a point x in Σ and the shortest path S in Σ from x to the boundary of Σ. If the length of S is L and the three-curvature along S is bounded below by R_0, it can be shown that

$$R_0 L^2 \leq 8\pi^2/3. \qquad (6)$$

This inequaltiy is applied by Schoen and Yau to the minimal-area two-surface which spans Γ to obtain (3).

One way of sharpening the Schoen and Yau result is to find a better measure for the size of Ω. Let me define $\mathcal{R}'(\Omega)$ as the size of the largest minimal-area two-surface that can be imbedded in Ω, where the size of a set is the distance from the boundary to that internal point which is furthest from the boundary. Inequality (6) gives us

$$R_0[\mathcal{R}'(\Omega)]^2 \leq 8\pi^2/3, \qquad (7)$$

just like (3).

The important reason (apart from computational ease) of switching from (3) to (7) is that one can show that for any set Ω,

$$\mathcal{R}'(\Omega) \geq \mathcal{R}(\Omega), \qquad (8)$$

and thus make an improvement in the inequality.

Consider an imbedded three-torus of radius p of the kind used to define $\mathcal{R}(\Omega)$. Now draw a closed curve Γ' on the surface of the torus "perpendicular" to the curve Γ which generated the torus, i.e., so that Γ and Γ' form a pair of linked rings. Consider the minimal-area surface that spans Γ'. This surface must cut Γ at some point P. This point P must be at least a distance p from the boundary of the part of the minimal surface contained in Ω. Hence

$$\mathcal{R}'(\Omega) \geq p,$$

and so (8) must be correct.

A natural arena to test estimate (7) is to consider a spherically symmetric region with positive constant scalar curvature. Such a region can be constructed by conformal transformation of flat space ($g_{ij} = \phi^4 \delta_{ij}$) with the following conformal factor:

$$\phi = \begin{cases} (1+\alpha r_0^2)^{3/2}(1+\alpha r^2)^{-1/2}, & r \leq r_0, \\ 1+\alpha r_0^3/r, & r > r_0, \end{cases} \tag{9}$$

where r is the flat-space radial coordinate, and α and r_0 are positive constants. We get for the scalar curvature

$$R = \begin{cases} 24\alpha(1+\alpha r_0^2)^{-6}, & r \leq r_0, \\ 0, & r > r_0. \end{cases} \tag{10}$$

Thus we have a spherically symmetric uniform density region matched to an exterior Schwarzschild solution.

The total mass can be read from the $1/r$ part of the conformal factor to give

$$M = 2\alpha r_0^3, \tag{11}$$

The proper surface area of any sphere of coordinate radius r is $4\pi\phi^4 r^2$. Thus the transformation from the flat radius r to the Schwarzschild-coordinate radius $\bar{r}$ is given by

$$\bar{r} = \phi^2 r, \tag{12}$$

and the Schwarzschild radius of the surface of the star is given by

$$\bar{r}_0 = (1+\alpha r_0^2)^2 r_0. \tag{13}$$

Now it is easy to see that the Schwarzschild condition $2M/\bar{r}_0 \leq 1$ reduces to

$$(1-\alpha r_0^2)^2 \geq 0 \tag{14}$$

and is satisfied for any choice of α and r_0. The limiting case is $\alpha r_0^2 = 1$, where we get $2M/\bar{r}_0 = 1$.

Any two-surface imbedded in a three-manifold has an induced two-metric g_{AB} and an induced two-extrinsic curvature k^{AB} ($A,B = 1,2$). Variations of a surface S spanning a fixed boundary can be generated by any scalar function h on the surface which vanishes on the boundary. The first variation of the area is given by

$$\delta A = -\int_S \sqrt{{}^{(2)}g}\, h \,\mathrm{tr}k \, d^2 s. \tag{15}$$

Since this has to vanish for every h, the surface is an extremal area surface if and only if

$$\mathrm{tr}k = g_{AB} k^{AB} \equiv 0. \tag{16}$$

If the surface is minimal, rather than just extremal, we also require that the second variation of the area be positive. The second variation is given by

$$\delta\delta A = -\int_S \sqrt{{}^{(2)}g}\, \{h\, {}^{(2)}\nabla^2 h + \tfrac{1}{2}({}^{(3)}R - {}^{(2)}R) h^2 + \tfrac{1}{2} k^{AB} k_{AB} h^2\} d^2 s \tag{17}$$

(assuming $\mathrm{tr}k = 0$), where ${}^{(2)}\nabla^2$ is the two-dimensional Laplacian and ${}^{(2)}R$ is the two-scalar curvature of S. One way of checking for area minimality is to find that function $\bar{h}$ which minimizes (17). Thus we vary h in (17) and get an eigenvalue equation for $\bar{h}$:

$$-{}^{(2)}\nabla^2 \bar{h} - C\bar{h} = \lambda \bar{h}, \quad \bar{h} = 0 \text{ on } \partial S, \tag{18}$$

where $C = \frac{1}{2}[{}^{(3)}R - {}^{(2)}R] + \frac{1}{2} k^{AB} k_{AB}$, and λ is a constant. The eigenvalue λ enters as a Lagrange multiplier in the problem because we wish to normalize $\bar{h}$, i.e., $\int \bar{h}^2 = 1$. The function $\bar{h}$ which minimizes (17) is the one which corresponds to the lowest eigenvalue λ_0 of (18) and the value of (17) is then just λ_0. Thus the surface is a minimal-area surface if all the eigenvalues of (18) are positive.

The obvious place to look for a minimal-area surface in the constant-density sphere is in the equatorial plane. It is obviously an extremal surface because $k^{AB} = 0$ for that surface and so clearly $\mathrm{tr}k \equiv 0$. For the equatorial plane the function C in (18) turns out to be a constant,

$$C = 8\alpha(1+\alpha r_0^2)^{-6}. \tag{19}$$

It is a straightforward exercise to show that the function

$$\tilde{h} = (1-\alpha r^2)/(1+\alpha r^2) \tag{20}$$

is a solution to

$$[-{}^{(2)}\nabla^2 - 8\alpha(1+\alpha r_0^2)^{-6}]\tilde{h} = 0 \tag{21}$$

on the equatorial plane. Clearly $\tilde{h} = 0$ at $r = \alpha^{-1/2}$ and so is a zero-eigenvalue solution to (18) when the boundary of S is the ring $r = \alpha^{-1/2}$. However, $\tilde{h}$ is positive everywhere inside $r = \alpha^{-1/2}$. This means that zero must be the lowest eigenvalue of (18) for this set ("ground states have no nodes").[4] Further, the lowest eigenvalue of any set enclosed by $r = \alpha^{-1/2}$ must be greater than zero and so must be a minimal-area surface, whereas the lowest eigenvalue of any set which includes $r = \alpha^{-1/2}$ must be negative and so therefore cannot be a minimal-area surface.[4]

Now the proper distance from the origin of coordinates to a ring at $r = \alpha^{-1/2}$ is

$$L = \int_0^{\alpha^{-1/2}} \phi^2 dr$$
$$= (1+\alpha r_0^2)^3 \alpha^{-1/2} \arctan(\alpha^{1/2} r)\,|_{r=0}^{r=\alpha^{-1/2}}$$
$$= \tfrac{1}{4}\pi \alpha^{-1/2}(1+\alpha r_0^2)^3. \tag{22}$$

Hence [since ${}^{(3)}R = 24\alpha(1+\alpha r_0^2)^{-6}$]

$$L^2 R_0 = 3\pi^2/2, \tag{23}$$

and thus we have a lower limit [to compare with (7)]

$$R_0[\mathcal{R}'(\Omega)]^2 \geq 3\pi^2/2. \tag{24}$$

What happens to the minimal surface for a ring larger than $r = \alpha^{-1/2}$? It turns out that the proper area of the plane of coordinate radius r_1 is

$$A_0(r_1) = (1+\alpha r_0^2)^6 \frac{\pi r_1^2}{1+\alpha r_1^2}, \tag{25}$$

whereas the area of the hemisphere of radius r_1 is

$$A_1(r_1) = (1+\alpha r_0^2)^6 \frac{2\pi r_1^2}{(1+\alpha r_1^2)^2}. \tag{26}$$

Now

$$A_0/A_1 = \tfrac{1}{2}(1+\alpha r_1^2). \tag{27}$$

Therefore if $r_1 < \alpha^{-1/2}$, $A_0 < A_1$, but if $r_1 > \alpha^{-1/2}$, $A_0 > A_1$. Thus the ring of radius $r = \alpha^{-1/2}$ supports two surfaces of equal area, the plane and the hemisphere, and the surface of minimum area which spans a ring of radius larger than $\alpha^{-1/2}$ is a prolate spheroid.

It turns out that the extrinsic curvature of the sphere of radius $r = \alpha^{-1/2}$ vanishes identically. Further, it turns out that the function C [in Eq. (18)] is identical to that for the plane,

$$C = 8\alpha(1+\alpha r_0^2)^{-6}. \tag{28}$$

Finally, it is possible to show that

$$\tilde{h} = \cos\theta \tag{29}$$

is a solution to (21) on the sphere. Thus, the hemisphere is also a minimal-area surface spanning the ring of radius $r = \alpha^{-1/2}$. The pole-to-ring distance for this minimal surface is

$$L = \frac{\pi}{2} r\phi^2 = \frac{\pi}{2} r(1+\alpha r_0^2)^3 (1+\alpha r^2)^{-1}$$
$$= \tfrac{1}{4}\pi\alpha^{-1/2}(1+\alpha r_0^2)^3. \tag{30}$$

Again we get [just as with (23) and (24)]

$$L^2 R_0 = 3\pi^2/2,$$
$$R_0[\mathcal{R}'(\Omega)]^2 \geq 3\pi^2/2. \tag{31}$$

When one checks the derivation of either (3) or (7) it becomes clear that they cannot be sharp estimates, i.e., there cannot exist configurations for which $R_0[\mathcal{R}'(\Omega)]^2 = 8\pi^2/3$. This means that there must exist a better constant, which clearly must lie somewhere between $3\pi^2/2$ and $8\pi^2/3$. I would like to conjecture that the lower limit I have obtained here ($3\pi^2/2$) is also the best that one can do and that the true inequality should read

$$R_0[\mathcal{R}'(\Omega)]^2 \leq 3\pi^2/2, \tag{32}$$

or

$$\rho_0[\mathcal{R}'(\Omega)]^2 \leq 3\pi/32. \tag{33}$$

This is based on the fact that the best estimate is invariably found in a highly symmetric situation. Obviously, a sphere of constant scalar curvature is exactly such a situation. Further, it is quite surprising that both evaluations (24) and (31) give exactly the same constant, and that the value is independent of r_0 (so long as $r_0 > \alpha^{-1/2}$).

Finally, let me return to the original Schoen and Yau estimate (3), based on the imbedded torus, and try to apply it to the uniform-density-sphere model. The proper distance from center to surface is [from (22)]

$$D = (1+\alpha r_0^2)^3 \alpha^{-1/2} \arctan(\alpha^{1/2} r_0). \tag{34}$$

We therefore would expect to imbed a torus of radius $D/2$ inside in the sphere. Now we get

$$R_0(D/2)^2 = 6[\arctan(\alpha^{1/2} r_0)]^2. \tag{35}$$

The maximum value of this occurs when $\alpha^{1/2} r_0$ gets large, in which case

$$\arctan(\alpha^{1/2} r_0) \to \pi/2$$

and

$$R_0(D/2)^2 \to 3\pi^2/2. \tag{36}$$

Therefore we expect a lower limit for (3)

$$R_0[\mathcal{R}(\Omega)]^2 \geq 3\pi^2/2, \tag{37}$$

just like (24) and (31).

Since we know that $\mathcal{R}'(\Omega) \geq \mathcal{R}(\Omega)$ for any set [Eq. (8)], if we believe (32) we must also accept

$$R_0[\mathcal{R}(\Omega)]^2 \leq 3\pi^2/2, \tag{38}$$

$$\rho_0[\mathcal{R}(\Omega)]^2 \leq 3\pi/32, \tag{39}$$

with the realization that these must now be sharp estimates, that no number smaller than $3\pi^2/2$ will do.

It cannot be overemphasized that these restrictions (33) and (39) on the size of stars are entirely independent of any equation of state for the material. Further, they are restrictions which motion in the star can only make more severe. If there is a matter current J^i present, the momentum constraint[2,3]

$$\nabla_j[K^{ij} - (\mathrm{tr}K)g^{ij}] = 8\pi J^i \tag{40}$$

means that we must have extrinsic curvature K^{ij} and thus increase the scalar curvature.

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