Critical Behavior of an Ising Spin-Glass

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Critical behavior of Ising spin-glass with $\pm J$ distribution is discussed. The results are based on the high-temperature series expansion of Edwards-Anderson susceptibility in powers of w ($=\tanh^2 J/kT$) for 2D, 3D, and 4D "cubic" lattices to orders 19, 17, and 15, respectively. The lengths of the series are comparable to the best available for the pure Ising model. While 3D and 4D systems exhibit transitions at finite temperatures, in 2D a zero-temperature transition with a power-law divergence (in inverse temperature) for the susceptibility is found. The estimates of critical temperatures and exponents are presented.

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Following the pioneering work of Edwards and Anderson[1] there has been a growing interest in understanding how the spins freeze into a randomly oriented spin-glass state in the language of phase transition.[2] By now the mean-field theory for the infinite-range model is well understood.[3] For short-ranged models various calculational techniques including high-temperature series expansion,[4] real-space renormalization group,[5] exact transfer-matrix calculations for finite-size systems,[6] defect scaling,[7] and numerical simulations[8,9] have been used. The predictions for the lower critical dimensionality (d_L) below which no transition occurs at finite temperatures have ranged from 2 to 4. In fact, for a long time, d_L was generally believed to be 4 for Ising spin-glasses. However, recent numerical simulations[8,9] in 3D systems strongly suggest a phase transition at a finite temperature.

The purpose of this paper is twofold. The first purpose is to report on an extensive high-temperature series expansion for Ising spin-glasses with $\pm J$ distribution. The estimates of the transition temperature and the exponent of the Edwards-Anderson susceptibility are presented for 2D, 3D, and 4D systems. While the results for 2D and 4D are new, for 3D similar results have been recently reported in numerical simulations.[8,9] The evidence is now overwhelming, both from series analysis and from numerical simulations, that the spin-glass ordering occurs at a finite temperature in 3D. The second purpose of this paper is to demonstrate that the series-expansion method is a valuable tool for understanding the critical behavior of disordered systems provided it is carried out to high orders and is analyzed with sufficient care. From our present work it is now evident why earlier attempts[4] to obtain information from high-temperature series led to incorrect results. The series were simply too short. Previously only the first ten terms in the power-series expansion in the variable w ($=\tanh^2 J/kT$) for the Edwards-Anderson susceptibility were known. Here, this expansion is carried out to nineteenth order in 2D, to seventeenth order in 3D, and to fifteenth order in 4D. Analogous series for the susceptibility of the pure Ising model are known to 21st order for 2D square lattices, to nineteenth order for 3D simple cubic lattices, and to seventeenth order for 4D hypercubic lattices[10] [The expansion parameter in this case is v ($=\tanh J/kT$).] We hope that this paper will stimulate further interest in the use of series-analysis techniques for disordered systems.

The Ising spin-glass is described by the Hamiltonian

$$-\beta\mathcal{H} = \sum_{(i,j)} J_{ij} S_i S_j. \tag{1}$$

Here the sum runs over each nearest-neighbor pair once. S_i is the Ising spin at site i which takes values ± 1. J_{ij} are independent random variables and with equal probability take values $\pm J$. The Edwards-Anderson susceptibility is given by

$$\chi_{SG} = N^{-1} \sum_i \sum_j [\langle S_i S_j \rangle^2]_J. \tag{2}$$

Here N is the total number of spins. Angular brackets refer to thermal averaging and the square brackets refer to averaging with respect to the distribution of J_{ij}.

An important ingredient of our calculation is the existence of the star-graph expansion (to be explained below) for χ_{SG}^{-1}; i.e., χ_{SG}^{-1} can be written explicitly as a sum over star graphs only.[11] That such an expansion holds for the free energy was proved by Ditzian and Kadanoff.[12] Our proof for χ_{SG}^{-1}, which will be discussed in the longer version of this paper,[13] is a generalization of the proof given by Rapaport[14] for the inverse susceptibility of the pure Ising model. In this paper we entirely concentrate on χ_{SG}. The series for the free energy will be given elsewhere.[13] Since the nonanalyticity of the free energy is rather weak, this series is difficult to analyze. We are currently looking into this problem.

We calculate χ_{SG} using a method similar to that used for the calculation of the susceptibility of the pure Ising model.[15] It can be shown[13] that

$$\chi_{SG}^{-1} = 1 + \sum_{g_s} L(g_s) * w(g_s). \tag{3}$$

Here the sum g_s is over all topologically distinct star graphs. A graph is said to have a point of articulation (k) if cutting all the bonds incident at k and removing the piece connected to k splits the graph into disconnected parts. A graph with no such points of articulation is called a star graph. $L(g_s)$ is called the lattice constant of the graph and is simply the number of ways per lattice site that the given graph can be embedded in the lattice. $W(g_s)$ is the weight of the graph.

In order to calculate the susceptibility series to order N one needs to carry out the summation in expression (3) for all star graphs with N or less bonds. In our calculation graphs were systematically enumerated with increasing cyclomatic number[15] and the lattice constants were evaluated following an algorithm due to Martin.[16] The weights were calculated by computation of the amplitude ψ_G, defined by (N_v is equal to the number of vertices in the graph)

$$\psi_G = \sum_{ij} (M^{-1})_{ij} - N_v, \tag{4}$$

for a graph and then the weights of all its subgraphs were subtracted off. Here the matrix M is defined by its elements M_{ij} given by $M_{ij} = [\langle s_i s_j \rangle^2]_J$. This method provides us with a large number of checks for the correctness of each weight. Since the weight of a graph with r bonds must start as w^r, all lower powers of w must cancel completely in the process of subgraph subtraction. Hence any small error would immediately show up. Unfortunately such an elegant internal check does not exist for the generation of the lattice constants. We checked their correctness by using them to calculate the susceptibility series for the Ising model. Our Ising-model series agree with those in the literature.[10] All computations were done on a RIDGE 32 minicomputer. Let us write the susceptibility series as

$$\chi_{SG} = 1 + \sum_n a_n w^n. \tag{5}$$

The coefficients a_n are given in Table I.

Here the first three terms (identical to the pure Ising model) contain little information on spin-glass ordering. In fact, one cannot hope to see any spin-glass behavior until one gets contributions from diagrams involving closed loops. This is because frustration is an essential feature of spin-glass and only occurs in closed loops. Hence any analysis which depends very sensitively on the first few terms is likely to give incorrect answers. It has been pointed out by Nickel[17] that the use of an Euler transform of the form $z = w/(1+bw)$ to go from a series in w to a series in z amounts to weighting all the higher-order coefficients with the lower-order ones. And with a larger b the higher-order terms in the original series are almost entirely suppressed. Although this procedure can give apparent convergence, one is using the early terms to estimate the critical behavior. We have avoided the use of Euler transforms entirely in our analysis.

We have found the method of first-order inhomogeneous differential approximants[18,19] most suitable to our analysis. We construct three polynomials P_1, P_2, and P_3 of order M,L, and J, respectively, such that $[P_1(0) = 1]$

$$P_1(w)(df/dw) + P_2(w) f(w) = P_3(w) + O(w^{M+L+J+2}), \tag{6}$$

where f is the function, with a power series in w, that is being represented. One can show that at points w_c where $P_1(w_c) = 0$ the solution to the differential equation has a singularity of the form $(w_c - w)^{-\gamma}$ with

TABLE I. Coefficients a_n for various cubic lattices.

n	2D	3D	4D
1	4	6	8
2	12	30	56
3	36	150	392
4	52	582	2408
5	116	2454	15 272
6	−108	6870	85 352
7	228	25 782	508 808
8	−2380	34 374	2 625 896
9	4084	202 486	15 111 976
10	−14 660	−323 730	72 067 672
11	80 052	3 428 262	421 464 680
12	−185 268	−8 217 746	1 851 603 192
13	877 428	110 253 462	11 810 583 208
14	−3 055 852	−241 502 106	46 346 625 320
15	9 445 156	2 495 638 934	347 729 503 368
16	−42 230 748	−12 217 497 930	
17	141 760 852	48 017 425 206	
18	−545 628 100		
19	2 140 276 820		

$\gamma = P_2(w_c)/P_1'(w_c)$. Note that the inhomogeneous term P_3 does not affect the critical behavior, and that P_3 only affects the coefficients of w^i in Eq. (6) with $i \leq J$. Hence P_1 and P_2 are entirely determined by a comparison of the coefficients of w^i with $J+1 \leq i \leq M+L+J+1$ in Eq. (6). Thus the critical behavior does not depend on the first J terms, making these approximants better suited to this problem. This was observed by Fisher and Au-Yang.[18]

For a series of given length a large number of approximants can be constructed corresponding to different values of M, L, and J. Not all of them are equally well behaved. For reasons of convergence Hunter and Baker[19] suggested the use of approximants in the neighborhood of $(M, L=M-2, J=M-2)$. We have looked at all the approximants and our results are based on those which show the best convergence. The following types of approximants are considered defective and hence are discarded: (i) approximants where a zero of P_2 comes very close to the physical singularity, giving rise to an anomalously small value of the exponent γ; (ii) approximants where a zero of P_1 occurs close to the origin on the positive real axis, hence hindering the integration of the differential equation in Eq. (6) from the origin to the critical region; (iii) approximants where a zero of P_1 and a zero of P_2 occur within a distance of 0.001 anywhere in the complex plane.

It should be remarked that the failure of certain approximants to give the correct global representation of the function does not necessarily imply that the critical parameters are incorrectly represented; however, to be on the safe side it is better to ignore such approximants.[20]

For the 3D series we construct all approximants with $4 \leq M \leq 7$, $2 \leq L \leq 5$, $2 \leq J \leq 6$. After discarding the defective approximants and restricting ourselves to those which use up terms in the series at least up to fifteenth order we are left with thirteen approximants. On this basis our estimate is (the uncertainty refers to the standard deviation) $\omega_c = 0.48 \pm 0.04$ ($T_c = 1.2 \pm 0.1$), $\gamma = 2.9 \pm 0.5$. This can be compared with the Monte Carlo data (Ogielski, Ref. 9) $T_c = 1.175 \pm 0.025$, $\gamma = 2.9 \pm 0.3$. In Fig. 1 we give a representative plot for χ_{SG}^{-1} obtained by solving the differential equation [Eq. (6)].

In 4D we consider all approximants with $4 \leq M \leq 9$, $2 \leq L \leq 6$, and $1 \leq J \leq 5$. Discarding the defective approximants and keeping only those which use up terms in the series at least up to thirteenth order we are left with 27 approximants. On this basis our result is[21] $\omega_c = 0.21 \pm 0.01$ ($T_c = 2.02 \pm 0.06$), $\gamma = 2.0 \pm 0.4$. Our results are in sharp contrast to the earlier work of Fisch and Harris,[4] who had predicted, on the basis of their ten-term series, that $\gamma \to \infty$ in 4D. Indeed, we find rather large values of γ if we consider only ten terms ($\gamma \cong 50$); however the approximants are mostly defective. An analysis of how the estimates rapidly settle down as a function of the order of the series will be discussed elsewhere.[13]

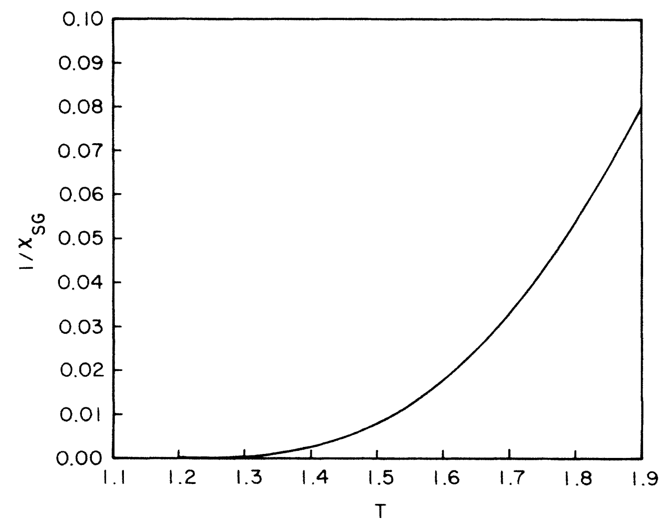

FIG. 1. A representative plot for χ_{SG}^{-1} in three dimensions obtained by integration of the ($L=5$, $M=6$, $J=3$) differential equation with the boundary condition $f(0)=1$. On the scale of the plot ten different approximants essentially coincide with the curve. The maximum deviation occurs near $T=1.5$ and is of the order of 0.0005.

A similar analysis in 2D shows no convergent singularity in the region of interest which we interpret as the absence of a finite-temperature transition or an essential singularity at zero temperature.[22] The variable w, then, is not the appropriate expansion variable as it has an essential singularity at $T=0$. Hence, we express the series in terms of a new variable z ($=1/T^2$). The method used for estimation of γ in this case is due to Baker, Rushbrooke, and Gilbert.[23] We expect that $(d\ln\chi/d\ln z)_{z\to\infty} = \frac{1}{2}\gamma$. We construct a power series in z for the expression $(d\ln\chi/d\ln z)$ and do a diagonal $[M/M]$ Padé to estimate γ. This method only uses series of even orders. To get estimates from series of odd orders we consider $A(z)$ ($=d\chi/dz$) instead of χ. Our best guess is $\gamma = 5.3$ (with an uncertainty of ± 0.3).

It is evident from our estimates that the uncertainties in various critical parameters are roughly a hundred times those for the corresponding estimates from a series of the same length for the pure Ising model. While in the Ising model the physical singularity is the closest to the origin, and determines the radius of convergence of the series, here it is not so, making it difficult to locate the critical point.

To conclude we would like to summarize our work. We have shown that Ising spin-glasses have a finite-temperature transition in 3D and 4D, and a zero-temperature transition in 2D. Clearly the conclusions drawn from the earlier series analysis were incorrect. We have also presented estimates of the critical exponent γ which characterizes the divergence of the Edwards-Anderson susceptibility of a spin-glass. Our value of γ for the 3D case is in excellent agreement with the recent results obtained from numerical simulations. The agreement is equally good for the transition temperature. Our results for 2D and 4D are, however, new.

From the point of view of series expansion we have shown that using the star-graph expansion it is possible to derive a series which is comparable in length to the best available series for the pure Ising model. This gives us the hope that the series expansion method can be a valuable tool to study other disordered systems. In this respect we have emphasized the fact that for the spin-glass problem it is very important to obtain a long series. Furthermore, since the series for disordered systems are likely to be noisier, it is important to develop sophisticated methods of analysis. We have discussed some of these concerns in the present context.

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