Unusual Bifurcation of Renormalization-Group Fixed Points for Interfacial Transitions

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Effective Hamiltonians for interfaces which arise, e.g., in the theory of wetting are studied by a nonlinear functional renormalization group exact in linear order and apparently accurate for all spatial dimensionalities, d. Two nontrivial fixed points are found for $d < 3$ which describe the critical manifold and the completely delocalized phase, respectively. As d varies, these do not bifurcate from the Gaussian fixed point at $d_u = 3$ but rather mutually annihilate leaving behind a line of unusual "drifting" fixed points. Correspondingly the critical exponents exhibit singular behavior as $d \to 3-$.

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Renormalization-group (RG) theory provides a general framework for our understanding of critical phenomena at phase transitions[1]: Critical exponents with classical/mean-field or nonclassical values are related to the presence of Gaussian or nontrivial RG fixed points, respectively. For typical bulk critical or multicritical phenomena, the appropriate nontrivial fixed point bifurcates from the Gaussian fixed point at the upper critical dimension d_u.[2] As a result of this type of bifurcation, the critical exponents are continuous, single-valued functions of the spatial dimension d, which can be studied by diagramatic perturbation expansions in $\varepsilon \equiv d_u - d$.

Here, we are concerned with critical phenomena associated with the unbinding (or depinning or delocalization) of interfaces or domain walls. Both wetting and commensurate-incommensurate transitions belong to this class of interfacial phase transitions.[3,4] For these problems, RG calculations have been confined, so far, to the vicinity of the Gaussian fixed point.[5–9] In the work reported here we introduce a new, nonlinear functional renormalization group for treating interfacial phenomena and, for the first time, find nontrivial fixed points. For interfaces subject only to thermal fluctuations, we locate *two* nontrivial fixed points for spatial dimensionalities $d < 3$ which bifurcate from a line of "drifting" fixed points when $d_u = 3$ rather than from the Gaussian fixed point as normally expected.[10] Correspondingly, the critical exponents exhibit unusual *singular* behavior as $d \to 3-$.

To proceed, consider two interfaces (or an interface and a wall) in d dimensions which, on average, are parallel to some $(d-1)$-dimensional reference plane with coordinate vector $\mathbf{x} = (x_1, \ldots, x_{d-1})$. The local interfacial separation will be denoted by $l(\mathbf{x})$. For the case in which only thermal fluctuations play a role, the interfacial configurations are governed by the effective Hamiltonian

$$\mathcal{H}\{l\} = \int d^{d-1}x \left[\tfrac{1}{2}\tilde{\Sigma}(\nabla l)^2 + V(l(\mathbf{x}))\right], \qquad (1)$$

in which $\tilde{\Sigma}$ is the interfacial stiffness. The interface potential $V(l)$ represents the Landau-Ginzburg free energy per unit area for uniform interface separation; it depends on the character of the underlying microscopic interactions.

Now, unbinding transitions described by (1) exhibit three distinct scaling regimes depending on d and the character of the microscopic interactions[3,4,9,11]: (i) a mean-field (MF) regime for large d and/or sufficiently long-ranged interactions; (ii) a weak-fluctuation regime with nonclassical exponents but the same (trivially determined) phase boundaries as in MF theory; and (iii) a strong-fluctuation (SFL) regime in which both exponents and phase boundaries are nontrivial. Here, we focus on the SFL regime which is characterized by[3,11]

$$V(l)l^{\tau} \to 0 \text{ as } l \to \infty \text{ with } \tau = 2(d-1)/(3-d) \qquad (2)$$

$(1 \le d \le 3)$. Within the functional RG procedure to be explained, we will see that, for fixed $d < 3$, the critical points for all $V(l)$ satisfying (2) map onto the *same* nontrivial fixed point potential, $V_c^*(l)$. Hence, the whole SFL regime is characterized by *universal* critical behavior. On the other hand, all potentials $V(l)$ satisfying (2) which lead to complete interface separation are mapped by the RG onto a second nontrivial fixed point, $V_0^*(l)$; this potential is purely repulsive, i.e., $V_0^*(l) > 0$ (all l) while the critical potential $V_c^*(l)$ has an attractive tail for large l: See Fig. 2 below.

The domains of attraction of both V_c^* and V_0^* lie within the subspace of interface potentials $V(l)$ satisfying (2). Within this subspace, we find *exactly one* relevant perturbation at $V_c^*(l)$ with scaling index $\lambda_1 > 0$, but only irrelevant perturbations at $V_0^*(l)$. The critical exponent $\nu_\parallel$ for the divergence of the parallel correlation length $\xi_\parallel$ at the critical unbinding transition is determined, as usual, by $\nu_\parallel = 1/\lambda_1$; the other exponents follow.[3,4,11,12]

Our RG calculations yield $\nu_\parallel = 2.04 \pm 0.05$ for $d = 2$ which compares very favorably with the exact value[12,13] $\nu_\parallel = 2$. This fact together with the exactness of the procedure to linear order in $V(l)$ (see below) indicates that our RG should be reliable and accurate for all d and, hence, be useful in analyzing other interface and membrane problems.[3,14] We also find that $\nu_\parallel$ *increases* with d and diverges as $d \to 3-$. This is unexpected since fluctuations are normally supposed to become less important as d increases leading to $\nu_\parallel$ *falling* to its mean-field value $\nu_\parallel = 1$. An excellent fit to the data for $2 \le d \le 2.975$ is provided by

$$\nu_\parallel \simeq \varepsilon^{-1/2}[\tfrac{1}{2}\ln(B/\varepsilon) + C\varepsilon]^{1/2}, \qquad (3)$$

with $\varepsilon = 3 - d \ge 0$, $B = 3$, and $C = 3.65$. Thus, as $d \to 3-$ we have "critical exponents for critical exponents" and, probably, logarithmic factors also! The changes in critical locus with d agree with standard expectations: The region of the phase diagram in which the interface remains bound increases continuously with d (see Fig. 1 below).

The results just summarized have been obtained from an approximate nonlinear functional RG which is an extension of Wilson's approximate integral recursion relation.[2,15] It acts in the space of functions, $V(l)$, which (i) vanish as $l \to +\infty$, and (ii) are large and positive for $l \to -\infty$. To write the recursion relation transparently, we introduce the energy-density and length scales

$$v \equiv k_B T \int_> d^{d-1}p, \quad \tilde{a}^2 \equiv \frac{k_B T}{\tilde{\Sigma}} \int_> \frac{d^{d-1}p}{p^2}, \qquad (4)$$

where $\int_> = (2\pi)^{1-d} \int_{\Lambda/b}^{\Lambda}$ in which Λ is the momentum cutoff implicitly embodied in (1) while[1] $b > 1$ is the usual spatial rescaling factor. Then the initial potential $V^{(0)}(l) \equiv V(l)$ is renormalized by successive applications of $V^{(N+1)}(l) = \mathcal{R}[V^{(N)}(l)]$ where[16]

$$\mathcal{R}[V(l)] \equiv -vb^{d-1} \ln\left[\int_{-\infty}^{\infty} \frac{dl'}{(2\pi)^{1/2}\tilde{a}} \exp\left(-\frac{l'^2}{2\tilde{a}^2} - \frac{1}{2v}[V(b^\zeta l - l') + V(b^\zeta l + l')]\right)\right], \qquad (5)$$

with $\zeta = \frac{1}{2}(3-d)$ (for $d \le 3$).[3,4]

Compared to Wilson's original method,[2,15] the new features of our RG are (a) the normalization of the integral in (5), which has been set to preserve the form of $V^{(N)}(l)$ for large l, as required for interface problems, and (b) the specific definition of the scale $\tilde{a}$, which was originally treated as arbitrary[2,15]: The choice (4) ensures that our RG is *exact* to linear order in V for all b and d.[16] In addition, (c) the "wave-function renormalization" embodied in (5), namely, $\mathbf{x} \to \mathbf{x}/b$, $l \to l/b^\zeta$ with $\zeta = \frac{1}{2}(3-d)$, is to be regarded as exact. The analogous transformation of the order parameter has been used previously for bulk critical phenomena, with $3-d$ replaced by $2-d$,[2,15] but in that case it has the significant drawback that the exponent η is erroneously forced to vanish.[1,2] However, for interface unbinding transitions η must, in fact,[3,4,11] vanish for all d, so that the chosen rescaling is correct!

The fixed point relation $V^*(l) = \mathcal{R}[V^*(l)]$ reflects the choice of the origin for l. It follows from (5) that a translation, $l \to l - \Delta l$, leads to

$$\mathcal{R}[V^*(l - \Delta l)] = V^*(l - \Delta l/b^\zeta). \qquad (6)$$

For $d < 3$, this implies the presence, at a fixed point, of an irrelevant (and, in fact, redundant) perturbation, $\partial V^*/\partial l$, with negative scaling index $\lambda_2 = -\frac{1}{2}\varepsilon$. At $d = 3$ this perturbation is evidently *marginal:* Then the existence of one fixed point would imply, via (6), a whole line of fixed points. However, such *stationary* fixed points can be ruled out in $d = 3$.[17] Instead, we find only a line of isomorphous *drifting* fixed points, $V_3^*(l)$, which satisfy

$$\mathcal{R}[V_3^*(l)] = V_3^*(l - \Delta l^*) \quad (d = 3), \qquad (7)$$

with $\Delta l^* > 0$; for $b = 2$ we get $\Delta l^* = 0.549(k_B T/\tilde{\Sigma})^{1/2}$.

It is convenient for computation to absorb the scale factors $\tilde{a}$ and v in (5) by writing $U(z) \equiv V(\sqrt{2}\tilde{a}z)/v$; the recursion relation for U is (5) with $v = 2\tilde{a}^2 = 1$, and, thus, no longer involves Λ. Now, initial potentials

$$U^{(0)}(z) = \begin{cases} -we^{-sz} + ue^{-2sz}, & \text{for } z > 0, \\ \infty, & \text{for } z < 0, \end{cases} \qquad (8)$$

with $s, u > 0$, are appropriate for systems in which all microscopic interactions are short ranged.[6,7] Note the presence of a *hard wall* which can be handled easily by our nonlinear RG but cannot be treated by the linear RG approaches[5-7,9] used previously. The control parameter w may be regarded as a temperature difference $T_c^{MF} - T$.

Numerical iterations starting with (8) reveal a two-

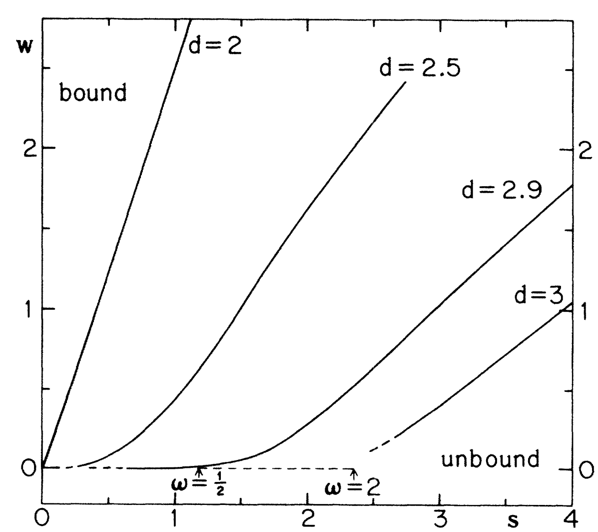

FIG. 1. Critical loci for the potential (8) with $u \equiv 1$ and various dimensionalities d (calculated with $b = 2$).

dimensional separatrix in the three-parameter space (s,w,u). The intersection of this separatrix with the plane $u=1$ is shown in Fig. 1 for various values of d. For fixed d, there is a locus $w=w_c(s)$ which divides the (s,w) plane into two parts. As a function of w in (8), the potentials $U^{(N)}$ for $N\to\infty$ exhibit the following three types of behavior: (i) For $w>w_c$, the minimum of $U^{(N)}$ becomes deeper and deeper: This corresponds to the *bound phase* in which the interfacial separation remains finite. (ii) For $w=w_c$, $U^{(0)}$ is eventually mapped onto the nontrivial fixed point $U_c^*(z)$ $[\equiv V_c^*(\sqrt{2}\tilde{a}z)/v]$ which governs the critical unbinding transition. (iii) For $w<w_c$, the attractive part of $U^{(N)}$ decays to zero, and $U^{(N)}$ is eventually mapped onto the purely repulsive fixed point $U_0^*(z)$ which governs the *completely unbound phase.*

The fixed point potentials, U_c^* and U_0^*, for $d=2$ are shown in Fig. 2. The behavior for general $d<3$ is similar, the decay to zero being always faster than exponential (and probably[17] as rapid as e^{-cz^2}). When $d\to 3-$ the location of the minimum diverges as

$$z^*(d)\approx A/\varepsilon \text{ with } A\simeq 3.38, \tag{9}$$

while its depth vanishes rapidly being well fitted by

$$U_{c,\min}^*\simeq -A_c\varepsilon^3/[\ln(B_c/\varepsilon)+C_c\varepsilon]^{1/2}, \tag{10}$$

with $A_c=\sqrt{51}$, $B_c=18.75$, and $C_c=19.02$ (for $b=2$). When $\varepsilon\to 0$ the repulsive part of the potentials U_c^* and U_0^* becomes increasingly close and merges at $d=3$. At the same time, the attractive part of the rescaled and shifted potential $\tilde{U}_c^*(z)=U_c^*(z^*+z)/|U_{c,\min}^*|$ appears to approach a well-defined function.[16]

The relevant scaling index λ_1 is found by studying the RG flows near the fixed point. If $U^{(N)}(z)=U_c^*(z)+E(z)$ with E small, iterations yield $U^{(N+n)}(z)\approx U_c^*(z)+b^{n\lambda_1}E(z)$; furthermore, the minima $U_{\min}^{(N+n)}$ are located close to z^*. Thus

$$\ln[(U_{\min}^{(N+2)}-U_{\min}^{(N+1)})/(U_{\min}^{(N+1)}-U_{\min}^{(N)})]/\ln b$$

serves as a good estimator for λ_1. As remarked, the value $\lambda_1\simeq 0.49$ found for $d=2$ is in good agreement with the exact value[12,13] $\lambda_1=\frac{1}{2}$; all the data are well represented by (3).

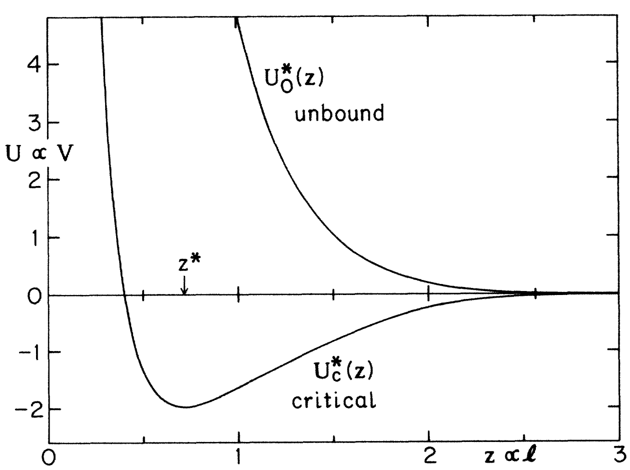

FIG. 2. The nontrivial fixed point potentials for $d=2$ (and $b=2$).

The numerical results described above were computed with a rescaling factor $b=2$. Since our nonlinear RG is not exact, one must ask about the dependence on b. Naturally, the fixed point potential itself does depend on b: E.g., for $d=2.975$, we find $U_{c,\min}^*\simeq -1.00\times10^{-4}$ and -1.29×10^{-4} and $z^*\simeq 129.13$ and 90.86 for $b=2$ and 4, respectively. On the other hand, the *forms* of the singularity in (9) and (10) seem independent of b.[16] Furthermore, the values of λ_1 for $b=2$ and $b=4$ differ by less than 3%. Thus, we expect the behavior for $\nu_\parallel$ given by (3) to hold for other values of b with only relatively small changes in the parameters.

The situation for hard-wall potentials (8) when $d=3$ is still a matter for debate. Within the linear RG, several scaling regimes arise depending on the parameter s[6,7,9]: Initially $w_c\equiv 0$ and $\nu_\parallel$ increases nonuniversally with $\omega=s^2/4\ln b$; but $\nu_\parallel$ becomes infinite for[7,9] $\omega\geq 2$, corresponding to an essential singularity as a function of $|T-T_c|$. In contrast, recent Monte Carlo simulations of wetting in an Ising model[18] see only critical behavior as predicted by mean-field theory.[19] The nonlinear RG presented here leads to the critical locus, $w=w_c(s)$, shown in Fig. 1 with w_c close to zero when $s_c=(8\ln 2)^{1/2}\simeq 2.35$ $(\omega=2)$. This is consistent with the result[7,9] of the linearized RG which gives $w_c\propto s-s_c\geq 0$. Along this locus the value $\nu_\parallel=\infty$ follows for $w_c>0$ by continuity with $d<3$: See (3). However, the precision of our numerical calculations is insufficient to decide whether w_c actually vanishes for $s\leq s_c$ or merely becomes exponentially small.

For $d<3$, we can study long-range perturbations at the fixed points. Thus consider a potential $V^{(0)}(l)=V^*(l)+E^{(0)}(l)$ with $E^{(0)}(l)\approx A^{(0)}/l^r$ as $l\to\infty$. One can see analytically that the flows around $V^*(l)$ yield $A^{(N)}\approx b^{N\lambda_r}A^{(0)}$ with

$$\lambda_r=d-1-\zeta r=\zeta(\tau-r), \tag{11}$$

with τ as in (2). As long as $r>\tau$, i.e., when $V^{(0)}$ belongs to the SFL regime, these perturbations are irrelevant. But if $E^{(0)}(l)$ represents an attractive tail with $r<\tau$, it is relevant and we recover the weak-fluctuation regime with $\nu_\parallel=1/\lambda_r$. Similarly the MF regime is characterized by an initial potential containing an attractive *and* a repulsive piece which are both relevant.

In summary, we have shown that the SFL regime in the unbinding of interfaces is governed by two nontrivial fixed points in a nonlinear functional renormalization group. These do not bifurcate from the Gaussian fixed point but rather from an unusual line of drifting fixed points [satisfying (7)]. Correspondingly, the critical exponents exhibit singular behavior as functions of d. Although we have focused on interfaces subject only to thermal fluctuations, the same approach and bifurcation mechanism should apply for wetting in random systems[3] and for the unbinding of membranes.[14]

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[12]Via $(2-\alpha)=(d-1)\nu_{\parallel}$, $\nu_{\perp}=\psi\equiv-\beta_s=\zeta\nu_{\parallel}$, $\zeta=\frac{1}{2}(3-d)$ $(d\geq 3)$.

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