Adverse Consequences of a Moving Vacuum-Plasma Boundary on Axisymmetric ac Helicity Injection

P. M. Bellan
California Institute of Technology, Pasadena, California 91125
(Received 18 June 1986)

The recent prediction of Liewer, Gould, and Bellan that a moving plasma-vacuum boundary significantly lowers the effectiveness of ac helicity injection is generalized by resolution of the apparent discrepancy between the helicity-conservation equations of Jensen and Chu and of Moffatt. It is shown that, if there are axisymmetric circular flux surfaces and a moving vacuum-plasma boundary, then the helicity injected by oscillating fields (if net injection occurs) is simply consumed by an increase in helicity dissipation due to the same oscillating fields.

PACS numbers: 52.55.Dy, 52.30.Jb, 52.35.Mw, 52.50.Gj

Magnetic helicity is a measure of the linking of magnetic flux tubes; this linking occurs whenever there are force-free currents (currents that flow along magnetic field lines). Bevir and Grey,[1] Schoenberg *et al.*,[2] and Jensen and Chu[3] suggested that a steady-state toroidal current could be generated by injection of magnetic helicity into a toroidal plasma; this would be accomplished by the modulation of both toroidal and poloidal fields in quadrature at very low frequencies ($\omega \ll \omega_{ci}$). This process has been interpreted by the author[4] as a beating between resistive diffusion and compressional Alfvén modes.

In Refs. 1–3 it was implicitly assumed that plasma filled the vacuum chamber right up to the wall throughout the modulation [cf. Fig. 1(a)]. Yet Ref. 4 showed that toroidal modulation should compress the plasma minor radius, so that the plasma should become separated from the wall, leaving a vacuum region behind [cf. Fig. 1(b)]. Recently, Liewer, Gould, and Bellan[5] investigated ac helicity injection with slab and axisymmetric plasma models in which the plasma-vacuum boundary moved and plasma was conserved during the modulation. They found that if the boundary motion is properly taken into account, then the driven current has a different and significantly smaller scaling than predicted by Refs. 1–4. Thus, the analysis of Liewer, Gould, and Bellan contradicts the predictions of the helicity-conservation equation of Jensen and Chu.

The purpose of this paper is to show that the essence of the contradiction lies in a difference between the helicity-conservation equation presented by Jensen and Chu,

$$dK/dt + \int d\mathbf{S}\cdot(\varphi\mathbf{B}+\mathbf{E}\times\mathbf{A}) = -2\int \eta\mathbf{J}\cdot\mathbf{B}\,d^3r, \quad (1)$$

and that presented by Moffat,[6]

$$dK/dt + \int d\mathbf{S}\cdot\mathbf{B}(\varphi-\mathbf{A}\cdot\mathbf{U}) = \eta\int d^3r(\mathbf{B}\cdot\nabla^2\mathbf{A}+\mathbf{A}\cdot\nabla^2\mathbf{B}). \quad (2)$$

Even though Eqs. (1) and (2) obviously differ, for both the helicity is defined as

$$K = \int \mathbf{A}\cdot\mathbf{B}\,d^3r. \quad (3)$$

This definition for helicity is gauge invariant provided that $\mathbf{B}\cdot d\mathbf{S}=0$, which is assumed both by Jensen and Chu (for ac helicity injection) and by Moffatt. With $\mathbf{B}\cdot d\mathbf{S}=0$ these equations become respectively

$$dK/dt + \int d\mathbf{S}\cdot\mathbf{E}\times\mathbf{A} = -2\int \eta\mathbf{J}\cdot\mathbf{B}\,d^3r, \quad (4)$$

and

$$dK/dt = \eta\int d^3r(\mathbf{B}\cdot\nabla^2\mathbf{A}+\mathbf{A}\cdot\nabla^2\mathbf{B}). \quad (5)$$

ac helicity injection[1-4] results from the inductive part $(-\partial\mathbf{A}/\partial t)\times\mathbf{A}$ of the term $\mathbf{E}\times\mathbf{A}$ in Eq. (4); this term does not exist in Eq. (5).

In this paper the discrepancy between the Jensen-Chu and the Moffatt equations will be resolved, and the following important results will be derived: (i) If there is a moving vacuum-plasma interface then no dc current can be driven in an axisymmetric plasma having circular flux surfaces. This is true despite the possibility of a net helicity injection because it is found that any net helicity injection is exactly balanced by an increase in helicity dissipation. (ii) If plasma nonconservation destroys the moving plasma-vacuum interface then dc currents can be driven as in Ref. 3. These results represent a generaliza-

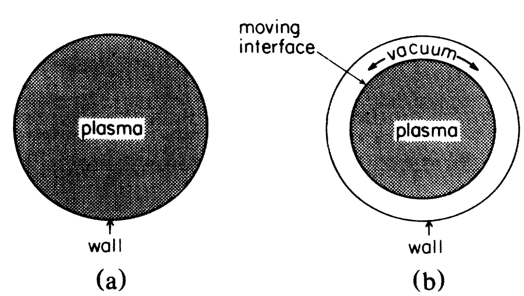

FIG. 1. (a) Plasma continuously filling up to its container so that there is no plasma-vacuum interface; (b) plasma with conservation of particles undergoing compression, so that a moving plasma-vacuum interface exists.

tion of the predictions of Ref. 5.

The discrepancy between Eqs. (1) and (2) occurs because the respective assumptions made by Jensen and Chu and by Moffatt are different. Specifically, Jensen and Chu manipulate Faraday's law to obtain a local conservation equation for helicity density,

$$(\partial/\partial t)\mathbf{A}\cdot\mathbf{B}+\nabla\cdot(\varphi\mathbf{B}+\mathbf{E}\times\mathbf{A})=-2\mathbf{E}\cdot\mathbf{B}, \tag{6}$$

then use the magnetohydrodynamic (MHD) Ohm's law $\mathbf{E}+\mathbf{U}\times\mathbf{B}=\eta\mathbf{J}$ to eliminate $\mathbf{E}$ from the right-hand side, obtaining

$$(\partial/\partial t)\mathbf{A}\cdot\mathbf{B}+\nabla\cdot(\varphi\mathbf{B}+\mathbf{E}\times\mathbf{A})=-2\eta\mathbf{J}\cdot\mathbf{B}, \tag{7}$$

and finally, integrate up to a fixed wall to obtain Eq. (1). The critical assumption here is that plasma exists right up to the wall *at all times* [cf. Fig. 1(a)] so that Ohm's law is *always* valid right up to the wall. [Note that the $\mathbf{U}\times\mathbf{B}$ term in Ohm's law is annihilated in this substitution, so that (i) plasma motion does not explicitly appear in Eq. (7), and (ii) any relation of the form $\mathbf{E}+\mathbf{Q}\times\mathbf{B}=\eta\mathbf{J}$, where $\mathbf{Q}$ is an arbitrary vector, would give the same result. For example, the nonMHD Ohm's law $\mathbf{E}-\mathbf{J}\times\mathbf{B}/ne=\eta\mathbf{J}$ relevant to the rotomak[7] would also give Eq. (7).]

Unlike Jensen and Chu, Moffatt does not assume that the plasma is enclosed by a vacuum chamber, but rather picks (in analogy to Ref. 5) the outer limit of his volume integral to be a surface which moves in such a way that the number of particles enclosed is constant. This surface then would be the plasma-vacuum boundary of an isolated plasma undergoing compression [Fig. 1(b)].

From a mathematical point of view, Jensen and Chu use the MHD Ohm's law right up to the wall and assume that $d^3r=$const so that

$$\int_{V_{\text{chamb}}} d^3r\frac{\partial}{\partial t}\mathbf{A}\cdot\mathbf{B}=\frac{d}{dt}\int_{V_{\text{chamb}}} d^3r\,\mathbf{A}\cdot\mathbf{B}, \tag{8}$$

whereas Moffatt (using Lagrangean variables) assumes that $d^3r\neq$const, but rather that $\rho d^3r=$const so that

$$\int_{V(t)}\rho d^3r\frac{\partial}{\partial t}\left[\frac{\mathbf{A}\cdot\mathbf{B}}{\rho}\right]=\frac{d}{dt}\int_{V(t)} d^3r\,\mathbf{A}\cdot\mathbf{B}. \tag{9}$$

In Eq. (8) V_{chamb} is the volume of the vacuum chamber (this volume is constant in time); in Eq. (9) $V(t)$ is the time-dependent volume of the plasma undergoing compression.

I will show here how to make the transition directly from Eq. (1) to Eq. (2) by appropriately changing Jensen and Chu's assumptions. This change is motivated by the fact that a compression of the plasma minor radius (as described in Ref. 4) should—if plasma is conserved—produce a vacuum layer between the plasma and the wall. In vacuum there are no currents, and so in vacuum it is not permitted to use the MHD Ohm's law to eliminate $\mathbf{E}$ in favor of $\eta\mathbf{J}$ as was done in going from Eq. (6) to Eq. (7). Instead, one may only use Eq. (7) in the plasma and then must revert to use of Eq. (6) for the vacuum layer between the plasma and the wall.

Thus, instead of integrating Eq. (7) over volume all the way up to the wall as in Ref. 3, one may integrate it only up to the plasma-vacuum boundary which, because of compression,[4] is moving. One obtains

$$\int_{V(t)} d^3r\frac{\partial}{\partial t}\mathbf{A}\cdot\mathbf{B}+\int_{S(t)} d\mathbf{S}\cdot(\varphi\mathbf{B}+\mathbf{E}\times\mathbf{A})=-2\int_{V(t)}\eta\mathbf{J}\cdot\mathbf{B}\,d^3r, \tag{10}$$

where all integrals are over the plasma volume *up to the moving boundary* $S(t)$. It is clearly not permissible to pull the partial time derivative out of the first term because the volume is changing.

For an arbitrary scalar ψ, the correct relation between a d/dt outside an integral having a moving boundary and a $\partial/\partial t$ inside is

$$\frac{d}{dt}\int_{V(t)} d^3r\,\psi=\int_{S(t)} d\mathbf{S}\cdot\mathbf{U}\psi+\int_{V(t)} d^3r\frac{\partial\psi}{\partial t}, \tag{11}$$

where $\mathbf{U}$ is the velocity of the boundary. Setting $\psi=\mathbf{A}\cdot\mathbf{B}$ gives

$$\frac{d}{dt}\int_{V(t)} d^3r\,\mathbf{A}\cdot\mathbf{B}=\int_{S(t)} d\mathbf{S}\cdot\mathbf{U}\mathbf{A}\cdot\mathbf{B}+\int_{V(t)} d^3r\frac{\partial}{\partial t}\mathbf{A}\cdot\mathbf{B}. \tag{12}$$

The integrand on the right-hand side of the first term can be rewritten as

$$\mathbf{U}(\mathbf{A}\cdot\mathbf{B})=\mathbf{A}\times(\mathbf{U}\times\mathbf{B})+\mathbf{B}(\mathbf{A}\cdot\mathbf{U})=\mathbf{A}\times\eta\mathbf{J}-\mathbf{A}\times\mathbf{E}+\mathbf{B}(\mathbf{A}\cdot\mathbf{U}), \tag{13}$$

where the MHD Ohm's law has been used to eliminate $\mathbf{U}\times\mathbf{B}$. Thus, the total time derivative of helicity is

$$\frac{d}{dt}\int_{V(t)} d^3r\,\mathbf{A}\cdot\mathbf{B}=\int_{S(t)} d\mathbf{S}\cdot[\mathbf{B}(\mathbf{A}\cdot\mathbf{U})+\mathbf{A}\times\eta\mathbf{J}-\mathbf{A}\times\mathbf{E}]+\int_{V(t)} d^3r\frac{\partial}{dt}\mathbf{A}\cdot\mathbf{B}, \tag{14}$$

which is significantly different from Eq. (8). If Eq. (14) is combined with Eq. (10), one obtains

$$\frac{d}{dt}\int_{V(t)} d^3r\,\mathbf{A}\cdot\mathbf{B}+\int_{S(t)} d\mathbf{S}\cdot[\mathbf{B}(\varphi-\mathbf{A}\cdot\mathbf{U})+\eta\mathbf{J}\times\mathbf{A}]=-2\int_{V(t)}\eta\mathbf{J}\cdot\mathbf{B}\,d^3r. \tag{15}$$

Finally, if we assume[6] that η=const and the gauge is such that $\nabla\cdot\mathbf{A}=0$, then Moffatt's Eq. (2) results. Of particular importance is the fact that the $-\mathbf{A}\times\mathbf{E}$ term of Eq. (14) cancels an identical term in Eq. (7). Since this term provided the ac helicity injection of Refs. 1–4, this analysis shows that if there is a moving plasma-vacuum boundary (such that plasma is conserved) then the term which produced ac helicity injection in Jensen and Chu's Eq. (1) is cancelled by an equal and opposite term associated with the moving plasma-vacuum boundary. It should be noted that the argument presented here affects only ac helicity injection; dc helicity injection[3] which occurs if $\int d\mathbf{S}\cdot\mathbf{B}\neq 0$ is allowed by both Eqs. (1) and (2), although there is a factor of 2 difference in the coefficient. Furthermore, there is now a new helicity flux in Eq. (15), namely $\eta\mathbf{J}\times\mathbf{A}$.

It is straightforward to evaluate this new helicity flux in toroidal geometry for the special case of axisymmetric circular flux surfaces. In this situation

$$B_r=0,\quad B_\theta=-\frac{1}{R+r\cos\theta}\frac{\partial}{\partial r}[(R+r\cos\theta)A_\varphi],$$

$$B_\varphi=r^{-1}\partial(rA_\theta)/\partial r,$$

and $\mathbf{U}=U\hat{\mathbf{r}}$. With these relations the MHD Ohm's law becomes

$$\eta\mathbf{J}=-\frac{\hat{\boldsymbol{\theta}}}{r}\frac{d}{dt}(rA_\theta)-\frac{\hat{\varphi}}{R+r\cos\theta}\frac{d}{dt}[(R+r\cos\theta)A_\varphi], \tag{16}$$

where $d/dt=\partial/\partial t+\mathbf{U}\cdot\nabla$. Thus, the helicity flux at the moving surface becomes

$$\int_{S(t)}d\mathbf{S}\cdot\eta\mathbf{J}\times\mathbf{A}=\int d\theta\int d\varphi\left\{rA_\theta\frac{d}{dt}[(R+r\cos\theta)A_\varphi]-(R+r\cos\theta)A_\varphi\frac{d}{dt}(rA_\theta)\right\}, \tag{17}$$

where the θ,φ integrals are from 0 to 2π and $d\mathbf{S}=\hat{\mathbf{r}}r\,d\theta(R+r\cos\theta)d\varphi$ has been used.

If the motion is such that $d(rA_\theta)/dt=0$ at the moving surface (i.e., conservation of total toroidal flux in the moving plasma, as assumed in Ref. 4 and in the slab model of Ref. 5) then

$$\left\langle\int d\mathbf{S}\cdot\eta\mathbf{J}\times\mathbf{A}\right\rangle=\int d\theta\int d\varphi(rA_\theta)\langle(d/dt)[(R+r\cos\theta)A_\varphi]\rangle=0,$$

where angular brackets denote time average over one period of oscillation in the moving frame. Similarly the helicity flux vanishes if $d[(R+r\cos\theta)A_\varphi]/dt=0$ (conservation of poloidal flux). These results are just a generalization of the results of Ref. 5.

Now suppose that neither total toroidal nor total poloidal fluxes are conserved during the oscillatory motion, but both are bounded, i.e., $d(rA_\theta)/dt\sim\sin\omega t$, $d[(R+r\cos\theta)A_\varphi]/dt\sim\cos\omega t$, where quadrature phasing has been chosen to provide a net helicity injection. The most general functions having this property are

$$rA_\theta=C_\theta(\xi)+D_\theta(\xi)\cos[\omega t-\alpha(\xi)],$$

$$(R+r\cos\theta)A_\varphi=C_\varphi(\xi)+D_\varphi(\xi)\sin[\omega t-\alpha(\xi)], \tag{18}$$

where $\xi(r,t)=r-\int_0^t U(r(t'),t')dt'$ is a constant of the motion (i.e., $d\xi/dt=0$). Inserting Eq. (18) in Eq. (17) and averaging over time gives

$$\left\langle\int_{S(t)}d\mathbf{S}\cdot\eta\mathbf{J}\times\mathbf{B}\right\rangle=\omega\int d\theta\int d\varphi D_\theta(r_S(0))D_\varphi(r_S(0)). \tag{19}$$

Here $r_S(t)$ is the position of the moving surface and use has been made of the relation $\xi(r_S(t),t)=r_S(0)$=const. Equation (19) gives a net helicity injection (unlike Ref. 5).

However, consider the time-averaged helicity dissipation as given by the right-hand side of Eq. (15):

$$\left\langle-2\int_{V(t)}\eta\mathbf{J}\cdot\mathbf{B}d^3r\right\rangle=2\left\langle\int_0^{r_S(0)}d\xi\int d\theta\int d\varphi\left\{\frac{\partial}{\partial\xi}(rA_\theta)\frac{d}{dt}[(R+r\cos\theta)A_\varphi]-\frac{\partial}{\partial\xi}[(R+r\cos\theta)A_\varphi]\frac{d}{dt}(rA_\theta)\right\}\right\rangle$$

$$=\omega\int d\theta\int d\varphi D_\theta(r_S(0))D\varphi(r_S(0)). \tag{20}$$

[Here both $D_\theta(0)=0$ and $\xi(0,t)=0$ have been used; these relations hold because both rA_θ and U vanish at $r=0$.] Equation (20) shows that the entire injected helicity [cf. Eq. (19)] is consumed by an increase in helicity dissipation due to the oscillating fields themselves. There is no uncommitted helicity flux remaining that could be used to drive dc currents and fields.

It is possible that this argument breaks down when there is a lack of axisymmetry and/or a velocity with nonzero curl. Also, in a real plasma the moving-boundary model is probably an excessive idealization because ionization and cross-field transport would tend to populate with new plasma the vacuum layer left behind by compression of the original plasma. The important quantity will be the rate at which resistivity (i.e., temperature) diffuses. If the new plasma filling up the vacuum layer is much colder than the plasma moved out by compression, then the incoming plasma will have a much larger resistivity and so not be much different from a vacuum. In this case, Eq. (15) would provide the correct description and so, as discussed by Liewer, Gould, and Bellan, ac helicity injection will not work as envisaged in Refs. 1–4. On the other hand, if the incoming plasma is the same temperature as the plasma moved out by compression, then there will effectively be uniform resistivity plasma right up to the wall at all times. In this case Jensen and Chu's assumption would be correct and ac helicity injection would work as predicted in Refs. 1–4.

It is a pleasure to acknowledge useful discussions with R. W. Gould, M. S. Schalit, and P. C. Liewer. This work was supported by the National Science Foundation under Grant No. ECS-8414541.

[1]M. K. Bevir and J. W. Grey, in Proceedings of the Reversed Field Pinch Workshop, Los Alamos Scientific Laboratory, New Mexico, 1981, edited by H. R. Lewis and R. A. Gerwin (unpublished); M. K. Bevir, C. G. Gimlett, and Guthrie Miller, Phys. Fluids **28**, 1826 (1985).

[2]K. F. Schoenberg *et al.*, Phys. Fluids **27**, 548 (1984).

[3]T. H. Jensen and M. S. Chu, Phys. Fluids **27**, 2881 (1984).

[4]P. M. Bellan, Phys. Rev. Lett. **54**, 1381 (1985).

[5]P. C. Liewer, R. W. Gould, and P. M. Bellan, "One-Dimensional Modeling of dc Current Drive by ac Helicity Injection in a Tokamak" (to be published).

[6]H. K. Moffatt, *Magnetic Field Generation in Electrically Conducting Fluids* (Cambridge Univ. Press, Cambridge, England, 1978), pp. 43–45.

[7]W. N. Hugrass *et al.*, Phys. Rev. Lett. **44**, 1676 (1980); M. L. Dutch, A. L. McCarthy, and R. G. Storer, Phys. Rev. Lett. **56**, 1563 (1986).

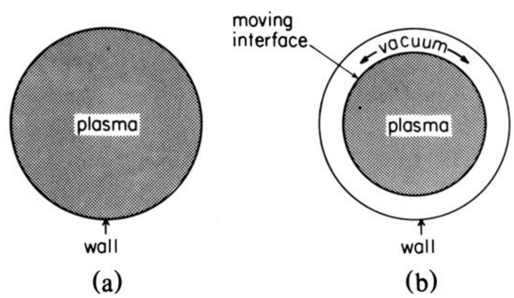

FIG. 1. (a) Plasma continuously filling up to its container so that there is no plasma-vacuum interface; (b) plasma with conservation of particles undergoing compression, so that a moving plasma-vacuum interface exists.