New Variables for Classical and Quantum Gravity

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A Hamiltonian formulation of general relativity based on certain spinorial variables is introduced. These variables simplify the constraints of general relativity considerably and enable one to imbed the constraint surface in the phase space of Einstein's theory into that of Yang-Mills theory. The imbedding suggests new ways of attacking a number of problems in both classical and quantum gravity. Some illustrative applications are discussed.

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Attempts at constructing perturbative quantum gravity have been unsuccessful. It is now generally believed that the problem lies in the basic assumption of the perturbation theory that the true space-time structure can be well approximated by a classical background geometry even below the Planck scale. From this standpoint, there is little hope in retaining the general perturbative framework and simply changing, e.g., the form of the Einstein Lagrangean by adding higher-derivative terms or supersymmetric matter. A more promising direction is to face the problem nonperturbatively. For, as has been emphasized by J. Klauder over the years, quantum gravity may well exist as an exact theory in spite of perturbative nonrenormalizability. The canonical quantization scheme provides a natural avenue in this direction since it does not require the fixation of a classical background geometry. Furthermore, the fact that the Hamiltonian structure of general relativity has certain essentially nonperturbative features—in the exact theory, the Hamiltonian is essentially given by the constraints, while the two decouple in any order in perturbation theory—suggests that qualitatively new results may arise from exact canonical quantization.

Over the years, the main obstacle to the canonical quantization program has come from the fact that the constraint equations have a complicated, nonpolynomial dependence on the traditional canonically conjugate variables. The purpose of this Letter is to report the existence of new variables in terms of which the constraints simplify considerably and to point out that the use of these variables provides new, nonperturbative approaches to problems in both classical and quantum gravity.

Let us begin with some mathematical preliminaries. Fix a three-manifold Σ and consider on it, in addition to tensor fields, $T^{a\cdots b}{}_{c\ldots d}$, fields with "internal" SU(2) indices, $\lambda^A, \mu_A, \ldots$. The SU(2) structure provides us with volume forms, ε^{AB} and ε_{AB}, on internal indices, satisfying $\varepsilon^{AB}\varepsilon_{AD}=\delta^B{}_D$. We shall use them to raise and lower these indices: $\lambda^A:=\varepsilon^{AB}\lambda_B$ and $\mu_A:=\mu^B\varepsilon_{BA}$. Now, given any isomorphism σ^{aA}_B between tangent vectors V^a and second-rank, trace-free, Hermitian fields $V^A{}_B$, the identification $V^a \equiv -\sigma^{aA}_B V^B{}_A$ solders the internal indices to the tangent space of Σ and makes them SU(2) spinor indices. Furthermore, each soldering form, σ, is a "square root" of a (positive definite) metric q_{ab} on Σ,

$$q_{ab}:=\sigma_a{}^{AB}\sigma_b{}^{MN}\varepsilon_{AM}\varepsilon_{BN}=-\mathrm{Tr}\,\sigma_a\sigma_b, \tag{1}$$

and singles out a unique torsion-free connection D on tensor and spinor fields on Σ satisfying

$$D_a\varepsilon_{AB}=0,\quad D_a\sigma^{bA}{}_B=0. \tag{2}$$

The configuration space $\mathscr{C}$ for general relativity is to be the space of all (suitably) regular and, if Σ is noncompact, asymptotically well-behaved soldering forms $\sigma^{aA}{}_B$. The phase space Γ is the cotangent bundle over $\mathscr{C}$. Thus, a point of Γ consists of a pair $(\sigma^{aA}{}_B, M_m{}^M{}_N)$, where M, a density of weight 1, is the momentum canonically conjugate to σ. The canonical variables (q_{ab}, p^{ab}) of the traditional Hamiltonian formulation[1] are now to be regarded as "derived" quantities: q_{ab} is given by (1) and p^{ab} by

$$p^{ab}:=-\mathrm{Tr}\,M_m\sigma^{(a}q^{b)m}\equiv M^{(ab)}. \tag{3}$$

As usual, not all points of Γ are accessible to the physical gravitational field: There are constraints. First, we have the familiar constraints

$$C^b(\sigma,M):=D_a p^{ab}=0, \tag{4}$$

$$C(\sigma,M):=(p^{ab}p_{ab}-\tfrac{1}{2}p^2)-\frac{(\det q)}{G^2}R=0, \tag{5}$$

where R is the scalar curvature of q_{ab}. However, since we have enlarged the configuration space from the space of the six-component fields q_{ab} to that of the nine-component fields $\sigma^{aA}{}_B$, we have three new constraints:

$$M^{[ab]}=0. \tag{6}$$

The canonical transformations generated by these constraints cause SU(2) rotations on the internal indices of σ and M.

The above framework is equivalent to the familar triad formalisms. The key new step is the introduction of new variables on Γ. Given any point (σ,M) of Γ, introduce two connections $^{\pm}\mathcal{D}$ on Σ:

$$^{\pm}\mathcal{D}_a \alpha_{bM} := D_a \alpha_{bM} \pm (i/\sqrt{2}) \Pi_{aM}{}^N \alpha_{bN}, \qquad (7)$$

where $\Pi_{aM}{}^N$ is given by[2]

$$\Pi_{aM}{}^N = G(\det q)^{-1/2}(M_{aM}{}^N - \tfrac{1}{2}\sigma_{aM}{}^N \sigma^b{}_{AB} M_b{}^{AB}). \qquad (8)$$

The use of these connections simplifies the structure of constraints (4) and (5) considerably. To see this, introduce connection one-forms $^{\pm}A_a$ and curvature two-forms $^{\pm}F_{ab}$ associated with $^{\pm}\mathcal{D}$:

$$^{\pm}\mathcal{D}_a \alpha_M =: \partial_a \alpha_M + G\ ^{\pm}A_{aM}{}^N \alpha_N, \qquad (9)$$

$$2\ ^{\pm}\mathcal{D}_{[a}\ ^{\pm}\mathcal{D}_{b]} \alpha_M =: G\ ^{\pm}F_{abM}{}^N \alpha_N, \qquad (10)$$

where ∂ is a fixed (c number) connection, also satisfying $\partial_a \varepsilon_{AB} = 0$. Then (6) can be rewritten as

$$^{\pm}\mathcal{D}_a \sigma^a{}_{AB} = 0, \qquad (6')$$

and, *modulo* (6), Eqs. (4) and (5) can be recast as

$$\mathrm{Tr}\sigma^a\ ^{\pm}F_{ab} = 0, \qquad (4')$$

$$\mathrm{Tr}\sigma^a \sigma^b\ ^{\pm}F_{ab} = 0. \qquad (5')$$

(Throughout, $\pm$ stands for $+$ *or* $-$; we can use either $^{+}A_a$ or $^{-}A_a$.) Constraints (4′)–(6′) are closed under the Poisson bracket and preserved by dynamics.

Note that the form of constraints (4′)–(6′) is simpler than that of (4)–(6) in at least two respects. First, (4′)–(6′) are at most quadratic in each of the new variables $(\sigma^a, {}^{\pm}A_a)$ while (4)–(6) involve nonpolynomial functions of q_{ab}. Second, if one were to regard $^{\pm}A_a$ as the new configuration variable and σ^a as the "momentum," (5′) involves only a "kinetic" term, quadratic in the new momenta, and is therefore structurally similar to the strong-coupling limit of (5) in which the "potential" term, R, is neglected. [This came about because $^{\pm}A_a$ knows both about p^{ab} and (the connection of) q_{ab}.] These features lead to new avenues especially in the quantum theory.

What is the physical interpretation of $^{\pm}A_a$? Consider a solution of g_{ab} of Einstein's equation obtained from initial data (σ^a, M_a) [satisfying (4)–(6)]. Then one has the following: $^{\pm}\mathcal{D}$ are the restrictions to Σ of the four-dimensional spin-connection ∇ on (un)primed $SL(2,C)$ spinors (e.g., $^{+}\mathcal{D}_a \lambda_M = q_a{}^b \nabla_b \lambda_m$), and $\mathrm{Tr}\ ^{\pm}F_{ab}\sigma^c \varepsilon^{abd} = (\sqrt{2}/G)(E^{cd} \mp iB^{cd})$, where E^{cd} and B^{cd} are the electric and magnetic parts, relative to Σ, of the Weyl tensor of g_{ab}. Thus, $^{\pm}A_a$ is a potential for the (anti-)self-dual part of the Weyl tensor. This fact leads to an interesting application: One can obtain (complex Lorentzian or, with minor convention changes, real Euclidean) self-dual solutions to Einstein's equation by simply setting $^{+}A_a = 0$. This *Ansatz* trivializes (4′) and (5′) and simplifies (6′) as well as the evolution equations (which, incidentally, automatically preserve the *Ansatz*). The resulting system of equations provides a new, simple, and convenient characterization of self-dual solutions.[3] Traditionally, H-space and twistor techniques have been used to study these solutions.[4] The new variables serve to bridge these techniques to the Hamiltonian methods. Other applications, to classical relativity, include the following: analysis of gravitational perturbations; interesting, exact solutions to constraint equations; understanding and generalizing of the results[5] on the relation between certain classes of solutions to Einstein's and Yang-Mills equations; and the use and role of hypersurface twistors in general relativity.

On the phase space Γ, the new variables have several interesting properties. Each of $\{^{+}A_a\}$, $\{^{-}A_a\}$, and $\{\tilde{\sigma}^a \equiv (\det q)^{1/2}\sigma^a\}$ forms a complete set of *commuting* variables with respect to the natural Poisson bracket on Γ. Furthermore, $^{\pm}A_a$ is "conjugate" to $\tilde{\sigma}^a$ in the sense that

$$\{^{\pm}A_m{}^{MN}(x), \tilde{\sigma}^a{}_{AB}(y)\}_{\mathrm{P.B.}} = \pm (i/\sqrt{2}) \delta_m{}^a \delta_A{}^{(M}\delta_B{}^{B)} \delta(x,y). \qquad (11)$$

[The factor of G in (9) ensures that $(A)\cdot(\tilde{\sigma})$ has dimensions of action.] *I shall therefore use* $\tilde{\sigma}^a$ *and* $^{\pm}A_a$ *as the basic variables.* [Note that one can replace σ^a by $\tilde{\sigma}^a$ in the constraints (4′)–(6′) free of charge.] These properties suggest identifications with certain variables that are featured in the Yang-Mills theory. $^{\pm}A_a$ is the connection one-form; $^{\pm}B^a := \varepsilon^{abc}\ ^{\pm}F_{bc}$, its magnetic field; and $\tilde{\sigma}^a$, the analog of the electric field E^a. In terms of these Yang-Mills variables, the constraints become

$$^{\pm}\mathcal{D}\cdot \mathbf{E} = 0, \qquad (6'')$$

$$\mathrm{Tr}\mathbf{E}\times\mathbf{B} = 0, \qquad (4'')$$

$$\mathrm{Tr}\mathbf{E}\cdot(\mathbf{E}\times\mathbf{B}) = 0. \qquad (5'')$$

Thus, every initial datum (satisfying constraints) (σ, M) for Einstein's equation yields initial data $(\mathbf{A}, \mathbf{E})$ for Yang-Mills equations which, in addition, satisfy four constraints which are purely algebraic in field strength; one has an imbedding of the Einstein constraint surface into the Yang-Mills theory. This imbedding preserves the Poisson-bracket structure of the two theories. On the other hand, it does not commute with time evolution; the Yang-Mills Hamiltonian is very different from Einstein's. However, since the Einstein Hamiltonian is a linear combination of constraints and a surface term, the simplification of constraints is significant also for Einstein dynamics.

To go over to quantum theory, as in the Yang-Mills case, we shall replace the basic variables $^{\pm}A_a$ and $\tilde{\sigma}^a \equiv E^a$ by operators $^{\pm}\hat{A}_a$ and $\hat{\sigma}^a$ satisfying the canonical commutation relation

$$[^{\pm}\hat{A}_m{}^{MN}(x), \hat{\sigma}^a{}_{AB}(y)] = (\hbar/\sqrt{2}) \delta_m{}^a \delta_A^{(M}\delta_B^{N)} \delta(x,y). \qquad (12)$$

The shift of variables from $(\hat{q},\hat{p})$ to $(\hat{\sigma},{}^{\pm}\hat{A})$ simplifies several issues in the quantum theory as a result of the features of constraints (4′)–(6′), noted below (5′). I now summarize the construction and the results that follow.

First, one can ask if there exists a factor ordering for the quantum version of constraints (4′)–(6′) for which the quantum constraints are closed under the commutator bracket, i.e., for which the evaluation of commutators yields a result in which a constraint operator always appears on the right. The answer is in the affirmative. Set

$$\hat{C}_N := (\sqrt{2}/\hbar)\int_\Sigma G^{-1}N_A{}^B({}^{\pm}\hat{\mathcal{D}}_a\hat{\sigma}^a)_B{}^A + N^a\mathrm{Tr}\hat{\sigma}^b\,{}^{\pm}\hat{F}_{ab} + N\,\mathrm{Tr}\hat{\sigma}^a\hat{\sigma}^b\,{}^{\pm}\hat{F}_{ab}, \tag{13}$$

where N stands for the triplet of smearing fields $(N^a{}_B,N^a,N)$. By use of (12) it then follows that

$$[\hat{C}_N,\hat{C}_M] = \hat{C}_P,$$

where

$$P_A{}^B = [N,M]_A{}^B + G^{-1}N^aM^b\,{}^{\pm}\hat{F}_{abA}{}^B + G^{-1}(MN^a - NM^a)(\hat{\sigma}^b{}_A{}^M\,{}^{\pm}\hat{F}_{abM}{}^B - \hat{\sigma}^b{}_M{}^B\,{}^{\pm}\hat{F}_{abA}{}^M),$$
$$P^a = \mathcal{L}_M N^a + 2(ND_bM - MD_bN)\hat{q}^{ab},\quad P = \mathcal{L}_M N - \mathcal{L}_N M. \tag{14}$$

In this result, the presence of internal indices and the consequent constraint (6′) plays a crucial role: Certain unwanted terms vanish because of the symmetry properties of their internal indices, and the commutators of (4′) and (5′) involve not only these constraints but also (6′). Also, the presence of the internal indices in (4′)–(6′) (as well as of $\hat{\sigma}^a$, the "square root" of $\hat{q}^{ab}$) makes it impossible to translate the preferred factor ordering in terms of the traditional variables $(\hat{q}_{ab},\hat{p}^{ab})$. Note, however, that the argument is only *formal;* I have not regularized the products of operator-valued distributions in (4′)–(6′). Nonetheless, formal closure is significant since it can fail even for systems with a finite number of degrees of freedom[6] where the issues of regularization never arise.

Next, we come to the issue of finding representations of the canonical commutation relation (12). Since, as noted above, the constraints are at worst quadratic in each of $\hat{\sigma}^a$ and $\hat{A}_a$, it is feasible to study both the $\tilde{\sigma}$ representation, in which quantum states are general complex-valued functionals of $\tilde{\sigma}$, and the ${}^{\pm}A_a$ representation, in which they are holomorphic functionals of ${}^{\pm}A_a$. [The analogs for a simple harmonic oscillator are respectively the position representation, $\psi \equiv \psi(x)$, and the Bargmann representation, $\psi \equiv \psi(z)$, $z = x + ip$.] This is in striking contrast to the situation with (q_{ab},p^{ab}) variables, where the momentum representation is unmanageable because the constraints have a complicated q dependence.

Let us focus on the ${}^{\pm}A_a$ representation since it enables one to borrow some ideas from (quantum) Yang-Mills theory. The weak-field limit has been studied in detail. Here, the quantum constraints are solved precisely by the requirement that the states be holomorphic functionals of the symmetric, trace-free, transverse part $(\delta A_a^i)^{STT}$ of the linearized connection $(\delta\,{}^{\pm}A_a)$, and the Hamiltonian is given simply by

$$H = (G/4\pi)\int_\Sigma (\delta A)^{STT}_{ab}(\delta A)^{\dagger ab}_{STT}\,d^3x \tag{15}$$

(which is analogous to the expression $H = ZZ^*$ for the simple harmonic oscillator). In exact theory the representation is being investigated by Jacobson and Smolin.[7]

Next, note that the constraints (4″)–(6″) in terms of the Yang-Mills variables $({}^{\pm}A_a,E^a)$ do not require a background structure such as a metric or a derivative operator. (This is to be contrasted with the Yang-Mills evolution equations which do require a background metric.) Consequently, one can take over techniques from the Hamiltonian lattice QCD[8] to put the quantum gravity on a lattice. The advantage of a lattice formulation is that the constraints do not have to be regularized. This line of investigation is being pursued by Renteln and Smolin.[9]

Finally, I have restricted the discussion to the vacuum case only for simplicity. It is straightforward to include a cosmological constant and matter sources—Yang-Mills sources fit in especially well—in the framework.

Details will appear elsewhere.

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[1]See, e.g., K. Kuchař, in *Quantum Gravity 2,* edited by C. J. Isham, R. Penrose, and D. W. Sciama (Oxford Univ. Press, New York, 1981).

[2]Note that, when (6) holds, Π_{ab} is just the extrinsic curvature.

[3]A. Ashtekar, T. Jacobson, and L. Smolin, to be published.

[4]See, e.g., M. Ko, M. Ludvigsen, E. T. Newman, and K. P. Tod, Phys. Rep. **71**, 51 (1981).

[5]R. S. Ward, "Integrable and Solvable Systems" (to be published).

[6]See, e.g., A. Ashtekar and M. Stillerman, J. Math. Phys.

(N.Y.) **27**, 1319 (1986).

[7]They construct quantum states from the holonomy operator of $^{\pm}A_a$ associated with closed loops and regularize the action of constraints on these loop states. Their results indicate that the metric q^{ab} would be degenerate microscopically and acquire its usual geometrical meaning only on macroscopic regions and in states with a large number of loops. (T. Jacobson and L. Smolin, private communication.)

[8]See, e.g., L. Susskind, in *Weak and Electromagnetic Interactions,* edited by R. Balian and C. H. Llewellyn Smith (North-Holland, Amsterdam, 1977).

[9]P. Renteln and L. Smolin, private communication.