Homogeneous Cosmological Models and New Inflation

Michael S. Turner and Lawrence M. Widrow
U. S. National Aeronautics and Space Administration/Fermilab Astrophysics Center, Fermi National Accelerator Laboratory, Batavia, Illinois 60510, and
Department of Physics, Enrico Fermi Institute, University of Chicago, Chicago, Illinois 60637
(Received 26 March 1986; revised manuscript received 4 August 1986)

The promise of the inflationary-Universe scenario is to free the present state of the Universe from extreme dependence upon initial data. Paradoxically, inflation is usually analyzed in the context of the homogeneous and isotropic Robertson-Walker cosmological models. We show that all but a small subset of the homogeneous models undergo inflation. Any initial anisotropy is so strongly damped that if sufficient inflation occurs to solve the flatness and horizon problems the Universe today would still be very isotropic.

PACS numbers: 98.80.Cq, 04.20.Jb, 98.80.Dr

The most attractive feature of the inflationary-Universe scenario[1] is that it offers the possibility of freeing the present state of the Universe, in regions as large as our present Hubble volume ($H^{-1} \simeq 10^{28}$ cm), from extreme sensitivity to the initial state of the Universe. Since it is very unlikely that we will ever be privy to the initial data for the Universe, this is indeed a very attractive attribute of inflation (for an alternative point of view, see Hartle and Hawking[2]). This extreme reliance upon initial data was emphasized by Collins and Hawking,[3] who demonstrated that without inflation the set of initial data which evolved into a model universe which resembles ours at the current epoch is of measure zero. While inflation holds the potential to free us from the initial data, for simplicity it is almost always analyzed within the context of an isotropic and homogeneous Robertson-Walker (RW) cosmological model. A key issue confronting inflation then is which subset of initial data for Einstein's equations undergo sufficient inflation to evolve to a model universe with large regions which resemble our present Hubble volume. Clearly not all the initial data do. A trivial counterexample is a very closed RW model which recollapses before it can inflate.

Here we consider all of the homogeneous models, the nine Bianchi classes, and the Kantowski-Sachs cosmology. (The Kantowski-Sachs cosmology is the exceptional case to the Bianchi classification of the homogeneous models; the topology of its spatial sections is $S^2 \times R$.) Wald[4] has shown that with the exception of a subset of Bianchi-IX models and, of the Kantowski-Sachs models, those which have very large, positive spatial curvature, all the homogeneous models with positive cosmological constant asymptotically evolve to de Sitter space. Of course, in inflationary-Universe models, the Universe does not in the strictest sense have a true cosmological term. Rather there is a vacuum energy density which depends upon an order parameter (usually the expectation value of some scalar field). So long as the scalar field is displaced from the zero-energy minimum of its potential and is slowly evolving, the vacuum energy is approximately constant and behaves like a cosmological term. The issue then is a dynamical one; does the Universe evolve into a de Sitter state before the scalar field reaches the minimum of its potential?

Steigman and Turner[5] have shown that anisotropy does not hasten the evolution of the scalar field, and so models which inflate in the absence of anisotropy also inflate in the presence of anisotropy, regardless of the amount of anisotropy present. Recently several authors[6] have raised the possibility that, even though inflation does occur in anisotropic models, if the Universe were intitally sufficiently anisotropic, growing modes of anisotropy might restore the anisotropy by the present epoch, thereby defeating the best efforts of inflation. This is the issue we will address in this Letter.[7]

We will show that while some homogeneous models will indeed again become very anisotropic, inflation postpones this event to an exponentially distant time in the future and models which inflate sufficiently to solve the horizon and flatness problems will today still be very isotropic. In this regard it has been known for a long while that inflation does not permanently render the Universe smooth within our Hubble volume; several authors have shown that if there were curvature perturbations (i.e., scalar density perturbations) present before inflation took place, then these perturbations would enter the horizon with the same amplitude as they would have in the absence of inflation, but at a much later time.[8,9] A finite epoch of inflation does not smooth the Universe globally; rather it creates large smooth patches, sufficiently large to encompass our Hubble volume at this late date in the history of the Universe.

Bianchi inflation.—For a detailed discussion of the Bianchi classification and models we refer the interested reader to Refs. 10–12 and for the Kantowski-Sachs cosmology to Ref. 13. We denote the scale factors of the principal axes of the Universe by X_i, $i=1$–3, the expansion rates in these directions by $h_i \equiv \dot{X}_i/X_i$, the proper volume of a unit comoving volume element by $V \equiv X_1X_2X_3$, and the mean expansion rate by $H \equiv \dot{R}/R \equiv \frac{1}{3}\dot{V}/V = (h_1+h_2+h_3)/3$, where $R \equiv V^{1/3}$ is the mean scale factor of the Universe. We will assume that the stress-energy in the Universe is described by a perfect, isotropic, and homogeneous fluid with energy density ρ and isotropic pressure $p=\gamma\rho$. Such a fluid with $\gamma=0$ corresponds to nonrelativistic matter, with $\gamma=\frac{1}{3}$ to a relativistic gas in thermal equilibrium, and with $\gamma=-1$ to vacuum energy. It follows from the conservation of stress-energy that

$$\rho \propto V^{-(1+\gamma)}. \tag{1}$$

In these space-times the equation of motion for a homogeneous scalar field ϕ with Lagrangean density $L=\frac{1}{2}\dot{\phi}^2 - V(\phi)$ is

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0, \tag{2}$$

where the overdot denotes a time derivative and the prime a derivative with respect to ϕ, and we use units where $\hbar = c = 8\pi G = 1$.

For all the Bianchi models except IV and VII_h the equations for the evolution of the scale factors can be written as[10,11,14]

$$\dot{h}_i + 3Hh_i = F_i + (\rho - p)/2, \tag{3}$$

$$\ddot{V}/V = F_1 + F_2 + F_3 + 3(\rho - p)/2, \tag{4}$$

where the F_i depend upon the scale factors X_i. For our purposes the crucial feature of the F_i is the fact that they decrease at least as fast as $V^{-2/3}$. That is the only property of the F_i that we will use. Physically, that corresponds to an effective energy density associated with anisotropy which decreases at least as fast as $V^{-2/3}$ (or R^{-2}), which means that the fastest-growing mode of anisotropy grows at the same rate as a curvature perturbation in a RW model. For Bianchi-I models the $F_i=0$; for Bianchi II, VI, VII, VIII, and IX the $F_i \propto V^{-2/3}$. For the Kantowski-Sachs cosmology there are only two independent scale factors: the radius of the two-sphere ($\equiv Y = Z$) and the scale factor of the flat dimension ($\equiv X$), and $F_x = 0$, $F_y = F_z = -Y^{-2}$.

For the Bianchi-IV and -VII_h models the equations cannot be put into diagonal form; there is an additional term on the right-hand side of Eq. (3), $G_i(\dot{h}_i, X_i)$ (see Refs. 11 and 12). The G_i decreases at least as fast as R^{-n} ($n \geq 1$). For simplicity we will not specifically consider these terms, as they do not qualitatively alter our analysis.

Except for the F_i terms in Eqs. (3) and (4), these equations are identical to those for the isotropic Robertson-Walker cosmology (where $X=Y=Z$, and $h_i = H$), and in the presence of stress-energy (with $p \neq \rho$) the different expansion rates rapidly become equal if the $F_i = 0$. For this reason we will use the F_i to quantify the degree of anisotropy. To this end we will denote the typical size of the F_i as F and will assume that $F \propto V^{-2/3}$, taking into account the most slowly decaying mode of anisotropy. We will denote the relative size of the anisotropy by

$$\epsilon \equiv F/\rho. \tag{5}$$

We will also assume that initially ϕ is displaced from the minimum of its potential (e.g., as a result of initial conditions or thermal effects), and that, in the *absence* of anisotropy during the time it takes ϕ to evolve to the minimum of its potential, the scale factor of the Universe grows by a factor of $\exp(N)$, i.e., the Universe inflates by N *e*-folds. We will not need to know the detailed evolution of ϕ here. We should emphasize that we are not addressing the question of the initial value of ϕ; we assume, as is done in the usual RW inflationary analysis, that ϕ is not initially at the zero-energy minimum of its potential. Indeed, one of the key unresolved issues confronting inflation is what determines the initial value of ϕ. For all viable models of inflation the scalar field ϕ is so weakly coupled that thermal effects do not serve to determine its value (see, e.g., Turner, Ref. 1).

For all but the subset of very highly positively curved Bianchi-IX models and Kantowski-Sachs models, the anisotropy increases the mean expansion, thereby increasing the $3H\dot{\phi}$ friction term in Eq. (2). For all but the aforementioned models the work of Wald[4] and Steigman and Turner[5] implies that the Universe will become dominated by vacuum energy (and become de Sitter type) before ϕ evolves appreciably from its initial value. Once the Universe is de Sitter type it will take ϕ the usual N Hubble times to evolve to the minimum of its potential, during which time the mean scale factor grows by a factor of $\exp(N)$.

Wald's result implies that asymptotically $\epsilon \to 0$; for our purposes we will denote the beginning of inflation to be the time when ϵ is sufficiently small, say $\epsilon_b \simeq 0.1$, so that F can be treated as a perturbation in Eqs. (3) and (4). During inflation we have

$$\rho \equiv \chi^2/3 \simeq V(\phi) \simeq M^4, \tag{6a}$$

$$\gamma \simeq -1, \tag{6b}$$

$$\ddot{V}/V \simeq \chi^2, \tag{6c}$$

$$V \propto \exp(\chi t), \tag{6d}$$

$$\dot{h}_i + \chi h_i = F + \chi^2/3, \tag{6e}$$

where as usual we have ignored the kinetic term $(\frac{1}{2}\dot{\phi}^2)$ as it is much smaller than $V(\phi)$ during inflation. Taking F to vary as $V^{-2/3} \propto \exp(-2\chi t/3)$ and solving for h_i and X_i to order ϵ we obtain

$$h_i = [1 + 3\epsilon_b \exp(-2\chi t/3)]\chi/3 + \text{const}\times\exp(-\chi t), \tag{7a}$$

$$X_i \propto \exp[\chi t/3 - 1.5\epsilon_b \exp(-2\chi t/3) - \text{const}\times\exp(-\chi t)/\chi], \tag{7b}$$

where "const" is an irrelevant integration constant.

As the Universe inflates, all the h_i approach the usual RW inflationary value of $\chi/3$ exponentially fast, and F decreases as $\exp(-2\chi/3t)$. Steigman and Turner[5] have shown that inflation will last at least the usual number of e-folds, and so at the end of inflation

$$\epsilon_e \leq \min[\epsilon_b, \epsilon_i]\exp(-2N), \tag{8}$$

since ρ remains constant during inflation and F decreases at least as fast as $V^{-2/3}$. Note that the value of ϵ_e is independent of the initial value of $\epsilon \equiv \epsilon_i$, provided that the initial value was larger than about 0.1 or so. If $\epsilon_i \leq 0.1$, then ϵ_e depends upon ϵ_i and is even smaller, $\epsilon_e \simeq \epsilon_i \exp(-2N)$. (Note that the exponentially rapid approach to de Sitter type also justifies the fact that the dynamics of inflation is usually analyzed in the context of a RW background).

Given ϵ at the end of inflation, how does the anisotropy of the Universe then evolve? To answer this we shall assume that after inflation $p = \gamma\rho$ with $\gamma \neq -1$ and that the Universe goes through three subsequent phases: a postinflation phase where the energy density is dominated by coherent oscillations of the ϕ field during which $\gamma = 0$; a radiation-dominated phase which begins when the ϕ particles decay, thereby reheating the Universe (to a temperature T_{RH}); and finally the current matter-dominated phase which begins when the Universe has a temperature of about 10 eV and is about 10^{10} sec old.

After inflation $\epsilon \ll 1$ so that the Universe is very nearly RW, and the F_i can be treated as small perturbations. If we work to lowest order in ϵ, and take the $F_i \propto V^{-2/3}$, it is simple to solve Eqs. (3) and (4):

$$\ddot{V}/V \simeq 3(1-\gamma)\rho/2 \Rightarrow V \propto t^{2/(1+\gamma)}, \tag{9}$$

$$\dot{h}_i + 2h_i t^{-1}/(1+\gamma) = F + (1-\gamma)\rho/2 \tag{10}$$

$$\Rightarrow h_i = [2t^{-1}/(3+3\gamma)]\{1 + [6\epsilon_0/(5+3\gamma)](t/t_0)^{(2+6\gamma)/(3+3\gamma)}\}, \tag{11}$$

where the subscript 0 refers to the value of the quantity at the reference time $t = t_0$. From Eq. (11) it is clear that ϵ grows as t (or R^2) during a radiation-dominated epoch and $t^{2/3}$ (or R) during a matter-dominated epoch. This is just as one would expect since the fastest-growing mode of anisotropy varies as R^{-2}, while $\rho_{rad} \propto R^{-4}$ and $\rho_{matter} \propto R^{-3}$.

We are now ready to compute the present ansitoropy in the expansion of the Universe. At the end of inflation ϵ is of order $\exp(-2N)$ or $\epsilon_i \exp(-2N)$ if the Universe was never dominated by anisotropy. While the energy density of the Universe is dominated by coherent oscillations, from $\rho \simeq M^4$ to $\rho \simeq T_{RH}^4$, ϵ grows as R or by a factor or $M^{4/3}/T_{RH}^{4/3}$. During the subsequent radiation-dominated phase, from $T \simeq T_{RH} - 10$ eV, ϵ grows as R^2, or by a factor of $10^{36}T_{10}$, where $T_{RH} = T_{10} \times (10^{10}$ GeV). Finally, in the current matter-dominated phase which began when the Universe had a temperature of about 10 eV, ϵ has grown by a factor of about 30 000. Bringing all of these factors together we find that the present level of anisotropy is at most

$$\epsilon_{today} \simeq \min[1, \epsilon_i]\exp(-2N)10^{46}M_{14}^{4/3}T_{10}^{2/3}. \tag{12}$$

The isotropy of the microwave background ($\delta T/T \leq 10^{-4}$) constrains ϵ_{today} to be less about 10^{-4}; sufficient inflation to guarantee this level of isotropy implies that

$$N \geq 57.5 + \ln(M_{14}^2 T_{10})/3, \tag{13}$$

which is precisely the amount of inflation required to solve the horizon and flatness problems (see Turner, in Ref. 1). This is not surprising as the fastest-growing mode of anisotropy varies as R^{-2}, just as the curvature of the Universe does. [For Bianchi IV and VII$_h$ the effect of the G_i terms modifies Eq. (13):

$$N \geq 243/n - 64 + (4/n - \tfrac{4}{3})\ln(M_{14}) + \ln(T_{10})/3.]$$

If there are growing modes of anisotropy, then the Universe will ultimately become very anisotropic again. If we assume that the Universe continues to be matter dominated and recall that $\epsilon \propto R \propto t^{2/3}$ when the Universe is matter dominated, it follows that ϵ will be of order unity when $t \simeq t_{anis}$, where

$$t_{anis} \simeq \min[1, \epsilon_i]^{-3/2} M_{14}^{-2} T_{10}^{-1} \exp(3N - 159)10^{10}\ \text{yr}. \tag{14}$$

Because of the interrelation between the isotropy and flatness problems, at about the same time, one would expect the Universe to become curvature dominated, i.e., Ω to deviate significantly from unity.

Summary.—We have shown that all the homogeneous models, except for the subset of Bianchi-IX and Kantowski-Sachs models which recollapse before they inflate, will undergo inflation and in the process become highly isotropic. As in the RW inflationary model of inflation, we have assumed that the scalar field responsible for inflation is initially displaced from the minimum of its potential. Regardless of the initial level of anisotropy, all models will be isotropic today provided that sufficient inflation occurred to solve the flatness and horizon problems. In the exponentially distant future the Universe may again become anisotropic provided that initially there were growing modes of anisotropy. As with the horizon and flatness problems, inflation merely postpones the inevitable.

Now that the homogeneous models seem to be in hand, the more difficult case of inhomogeneous models must be addressed. In this regard, it has already been shown that small inhomogeneities are no obstacle to inflation,[8,9] and recently the analogue of Wald's result[4] for inhomogeneous models has been proven.[15,16]

We gratefully acknowledge conversations with L. Jensen and J. Stein-Shabes, who have done similar independent work,[17] and J. Barrow's critical reading of the manuscript. This research was supported by the U. S. National Aeronautics and Space Administration (at Fermilab), the U.S. Department of Energy (at Chicago and at Fermilab), and by an Alfred P. Sloan Foundation Fellowship for one of us (M.S.T.).

[1]A. Guth, Phys. Rev. D **23**, 347 (1981); A. Linde, Phys. Lett. **108B**, 389 (1982); A. Albrecht and P. J. Steinhardt, Phys. Rev. Lett. **48**, 1220 (1982). For a recent review of inflation, see M. S. Turner, in *Proceedings of the Cargese School on Fundamental Physics and Cosmology,* edited by J. Audouze and J. Tran Thanh Van (Editions Frontières, Gif-sur-Yvette, France, 1985).

[2]J. Hartle and S. W. Hawking have advocated the quantum-cosmology approach to address the question of the initial state of the Universe and with certain assumptions about allowed spacetimes they can specify the initial state by calculating the wave function of the Universe; see J. Hartle and S. W. Hawking, Phys. Rev. D **28**, 2960 (1983). We also refer the interested reader to the recent review of quantum cosmology by A. Vilenkin, in *Proceedings of the Thessaloniki Conference on Particles and Cosmology, Thessaloniki, Greece, June 1985,* edited by G. Lazarides and Q. Shafi (North-Holland, Amsterdam, 1986).

[3]C. B. Collins and S. W. Hawking, Astrophys. J. **180**, 317 (1973).

[4]R. M. Wald, Phys. Rev. D **28**, 2118 (1983); Weber has explicitly generalized Wald's result to the Kantowski-Sachs models [E. Weber, J. Math. Phys. (N.Y.) **25**, 3279 (1984)].

[5]G. Steigman and M. S. Turner, Phys. Lett. **128B**, 295 (1983).

[6]Among others, this concern has been voiced by J. D. Barrow and D. H. Sonoda, Phys. Rep. **139**, 1 (1986), footnote on p. 45; and J. Hartle, University of California, Santa Barbara, Institute for Theoretical Physics, Report No. NSF-ITP-86-80, 1986 (to be published).

[7]Some aspects of this issue have also been addressed by Ø. Grøn, Phys. Rev. D **32**, 2522 (1985); B. C. Paul, D. P. Datta, and S. Mukherjee, Mod. Phys. Lett. **A1**, 149 (1986); I. Moss and V. Sahni, Phys. Lett. B (to be published).

[8]J. M. Bardeen, P. J. Steinhardt, and M. S. Turner, Phys. Rev. D **28**, 679 (1983).

[9]J. A. Frieman and M. S. Turner, Phys. Rev. D **30**, 265 (1984); R. Brandenberger and R. Kahn, Phys. Rev. D **29**, 2172 (1983).

[10]G. F. R. Ellis and M. A. H. MacCallum, Commun. Math. Phys. **12**, 108 (1969); M. P. Ryan and L. C. Shepley, *Homogeneous, Relativistic Cosmologies* (Princeton Univ. Press, Princeton, 1975).

[11]H.-H. von Borzeskowski and V. Mueller, Ann. Phys. (Leipzig) **7**, 361 (1978).

[12]See, e.g., A. Harvey and D. Tsoubelis, Phys. Rev. D **15**, 2734 (1977). It is interesting to note that these authors have constructed a rather general Bianchi-IV solution which does not permit a cosmological term. If this is a feature generic to Bianchi IV, then it would imply that, if nonzero vacuum energy is a part of the microphysics of our Universe, Bianchi-IV models are of no cosmological relevance to us.

[13]A. S. Kompaneets and A. S. Chernov, Zh. Eksp. Teor. Fiz. **47**, 1939 (1964) [Sov. Phys. JETP **20**, 1303 (1965)]; R. Kantowski and R. K. Sachs, J. Math. Phys. (N.Y.) **18**, 2116 (1979); C. Collins, J. Math. Phys. (N.Y.) **18**, 2116 (1979).

[14]Strictly speaking, Eqs. (3) and (4) are only valid in the case that the flow of the stress-energy is orthogonal to the homogeneous hypersurfaces. Before and during inflation the only relevant stress-energy present is that of the scalar field, whose flow is necessarily orthogonal to the homogeneous hypersurfaces, and after inflation we tacitly make the seemingly reasonable assumption that the radiation stress-energy produced by the decay of the vacuum energy during reheating has a four-velocity which is orthogonal to the homogeneous hypersurface. Thus Eqs. (3) and (4) are applicable throughout.

[15]A. A. Starobinskii, Piz'ma Zh. Eksp. Teor. Fiz. **37**, 55 (1983) [JETP Lett. **37**, 66 (1983)].

[16]L. Jensen and J. Stein-Schabes (to be published).

[17]L. Jensen and J. Stein-Schabes, Phys. Rev. D **34**, 931 (1986).