Large Intensity Fluctuations for Wave Propagation in Random Media

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(Received 14 July 1986)

The intensity pattern generated by a monochromatic point source in a random medium is studied. The intensity-intensity correlation function is calculated and it is shown that the intensity, as a function of coordinate, exhibits large fluctuations (the speckle pattern). The sensitivity of this speckle pattern to small changes in the source frequency is also studied.

PACS numbers: 42.20.Ji, 71.55.Jv

A wave propagating in a random medium undergoes multiple scattering from the inhomogeneities. The scattered waves interfere with each other and, as a result, a certain intensity pattern is formed. In a random medium, as opposed, e.g., to a crystal, one would, naively, expect an efficient averaging process and therefore a fairly smooth intensity pattern, with only small intensity fluctuations. Instead, however, one finds a highly irregular pattern, with large intensity changes over short distances. The irregularities are not due to noise. Each microscopic realization of the random medium, i.e., each sample of the statistical ensemble, displays its own pattern—a "fingerprint" which reflects the specific arrangement of the inhomogeneities (impurities) in that sample. This phenomenon is quite familiar in optics where it is termed "a speckle pattern" and usually refers to an intensity pattern formed on a screen by light reflected from a rough surface. Below, this term is used in a somewhat broader sense and refers to an intensity pattern formed in the bulk of a disordered medium when a wave (electromagnetic, acoustic, or an electron wave) propagates through it.

There exists a huge literature on the subject.[1-3] In early work, usually certain assumptions were made directly on the statistics of the scattered light (rather than on the statistical properties of the disordered medium). The "first principles" work, i.e., that which tries to derive properties of the speckle patterns directly from the wave equation, is mostly limited to smooth inhomogeneities (the wavelength much shorter than the characteristic inhomogeneity size).[2-4] The subject of light propagation in random media has been recently given a new boost as a result of a number of experiments. These experiments revealed an enhanced backscattering,[5] in combination with large intensity fluctuations[6] and high sensitivity of the speckle pattern to relatively small changes of the source frequency.[7]

Similar phenomena exist, and are being extensively studied, in the electron transport in disordered systems. The point is that as long as the sample size is smaller than the inelastic scattering length (the mesoscopic regime), an electron propagates coherently through the entire sample and, thus, takes a "fingerprint" of the specific, for that sample, impurity arrangement. This manifests itself in various interference phenomena and in extreme sensitivity of the conductance to small changes of various factors.[8-12]

The purpose of the present work is to calculate some properties of speckle patterns, specifically the intensity correlation function, for a scalar field. Such a field can represent an electron wave function, an acoustic wave, or (if polarization effects can be neglected) one of the components of an electromagnetic wave. Consider, thus, an infinite disordered medium with a monochromatic point source (a transmitter) located at a point $\mathbf{r}_0$. The field at point $\mathbf{r}$ is detected by a receiver, and it is given by the Green's function which satisfies the wave equation

$$\{\nabla^2 + k_0^2[1+\mu(\mathbf{r})] + i\eta\} G_\omega(\mathbf{r},\mathbf{r}_0) = \delta(\mathbf{r}-\mathbf{r}_0). \quad (1)$$

Here $\mu(\mathbf{r})$ represents the fluctuating part of the refraction index or, in quantum mechanics, the (properly normalized) random potential. ω is the source frequency, η is a positive infinitesimal, and $k_0 \equiv \omega/c_0$ is the wave number in free space, c_0 being the corresponding speed of propagation (with an obvious change in wording for the Schrödinger-equation case). In fact, more generally, k_0 refers not to free space but rather to the average medium; i.e., the average value of μ is set equal to zero. To specify the problem completely one needs to give the statistical properties of the fluctuating refraction index $\mu(\mathbf{r})$. We consider white-noise Gaussian statistics, i.e.,

$$\langle \mu(\mathbf{r}) \rangle = 0, \quad \langle \mu(\mathbf{r})\mu(\mathbf{r}') \rangle = u\delta(\mathbf{r}-\mathbf{r}'), \quad (2)$$

where u is a constant and the angular brackets mean an average over the statistical ensemble of random samples.

The intensity $I_\omega(\mathbf{r},\mathbf{r}_0)$ at a point $\mathbf{r}$ is proportional to $|G_\omega(\mathbf{r},\mathbf{r}_0)|^2$. The proportionality coefficient depends on the strength of the source and on the intensity units, and is set equal to unity (the actual intensity can always be obtained by multiplying the results by the corresponding factor). Our purpose, thus, is to calculate the intensity-intensity correlation function:

$$C(\mathbf{r}-\mathbf{r}_0,\mathbf{r}'-\mathbf{r}_0;\Delta\omega)\equiv\langle I_\omega(\mathbf{r},\mathbf{r}_0)I_{\omega+\Delta\omega}(\mathbf{r}',\mathbf{r}_0)\rangle-\langle I_\omega(\mathbf{r}-\mathbf{r}_0)\rangle\langle I_{\omega+\Delta\omega}(\mathbf{r}'-\mathbf{r}_0)\rangle. \quad (3)$$

For $\Delta\omega=0$, C is expected to decay when $\mathbf{r}'$ moves away from $\mathbf{r}$. The corresponding decay length gives, in fact, the typical size of a speckle, i.e., of a bright or dark spot (in optical terms). Similarly, for $\mathbf{r}=\mathbf{r}'$ but $\Delta\omega\neq0$, C describes the sensitivity of the speckle pattern to a change in frequency.

The calculation is more conveniently done in momentum space; i.e., the Green's function is expanded in a Fourier series. The Fourier transform, $G_\omega(\mathbf{k},\mathbf{k}')$ (the same symbol is used to avoid cluttering up the notation), satisfies the integral equation

$$G_\omega(\mathbf{k},\mathbf{k}')=G_\omega^0(\mathbf{k})\delta_{\mathbf{k}\mathbf{k}'}-k_0^2\sum_{\mathbf{q}}G_\omega^0(\mathbf{k})\mu(\mathbf{q})G_\omega(\mathbf{k}-\mathbf{q},\mathbf{k}'), \quad (4)$$

where $\mu(\mathbf{q})$ is the Fourier transform of $\mu(\mathbf{r})$ and $G_\omega^0(\mathbf{k})=k_0^2-k^2+i\eta)^{-1}$ is the unperturbed Green's function. Averaging is done by the standard perturbation technique (e.g., see, Abrikosov, Gorkov, and Dzyaloshinski[13] for electrons, and Frish[14] for classical waves). $G_\omega(\mathbf{k},\mathbf{k}')$ is expanded in a perturbation series with respect to $\mu(\mathbf{q})$. Each term in the expansion is represented by a diagram, like that in Fig. 1(a), with solid and dashed lines corresponding to free propagation and scattering, respectively. Averaging then amounts to pairing the dashed lines in all possible ways. Products of two or four Green's functions, required for calculating the average intensity and the intensity correlation function, respectively, are handled in a similar way. Assuming weak disorder, one can then compute various quantities in perturbation theory with respect to the small parameter $1/k_0l$, where $l\equiv4\pi/uk_0^4$ is the elastic mean free path. In the leading approximation, one obtains for the averaged Green's function $\langle G_\omega(\mathbf{k},\mathbf{k}')\rangle=G_\omega(\mathbf{k})\delta_{\mathbf{k},\mathbf{k}'}$, with

$$G_\omega(\mathbf{k})=[k_0^2-k^2+i(k_0/l)]^{-1}, \quad (5)$$

which, after transformation back to real space, gives an exponential decay for $\langle G_\omega(\mathbf{r}-\mathbf{r}_0)\rangle$.

The average intensity at point $\mathbf{r}$ is given by

$$\langle I(\mathbf{r}-\mathbf{r}_0)\rangle=\Omega^{-2}\sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4}\exp[i(\mathbf{k}_1-\mathbf{k}_3)\cdot(\mathbf{r}-\mathbf{r}_0)]\langle G(\mathbf{k}_1,\mathbf{k}_2)G^*(\mathbf{k}_3,\mathbf{k}_4)\rangle, \quad (6)$$

where Ω is a normalizing volume. In the leading approximation, one has to sum the ladder diagrams, i.e., to consider the object $W_{\mathbf{k}\mathbf{k}'}(\mathbf{q},\Delta\omega)$ shown in Fig. 1(b) and known as "diffusion" in electron transport theory.[15] (Although for the intensity calculation it suffices to consider $\Delta\omega=0$, a finite $\Delta\omega$ is kept for further use.) Thick lines between the dashed lines of the ladder represent dressed propagators [Eq. (5)] and their complex conjugates (the lower lines). A factor uk_0^4/Ω is assigned to each dashed line. When the resulting geometric series is summed one obtains, for small q and $\Delta\omega$,

$$W_{\mathbf{k}\mathbf{k}'}(\mathbf{q},\Delta\omega)=\frac{12\pi}{\Omega l^3}\frac{1}{q^2-3i(\Delta\omega/lc_0)}. \quad (7)$$

"Small $\Delta\omega$" means $l\Delta\omega/c_0\ll1$, whereas "small q" means $ql\ll1$. This is sufficient if one is interested in $|\mathbf{r}-\mathbf{r}_0|\gg1$; i.e., the transmitter and receiver are separated by at least several mean free paths. To complete the intensity calculation one has to attach to the "box" in Fig. 1(b) four (dressed) propagators, with corresponding momenta, multiply the diagram by $\exp(i\mathbf{q}\cdot\mathbf{r})$ (the source location, $\mathbf{r}_0$, is taken to be at $\mathbf{r}=0$), and integrate over the momenta. The result is

$$\langle I(\mathbf{r})\rangle=3/16\pi^2lr, \quad (8)$$

which, up to a numerical factor due to different normalization, coincides with the result obtained in Ishimaru's book[1] by a different method. The $1/r$ dependence in Eq. (8) means that, for $r\gg l$, the wave propagates by a diffusion process and the intensity obeys a diffusion equation (see e.g., Ref. 1).

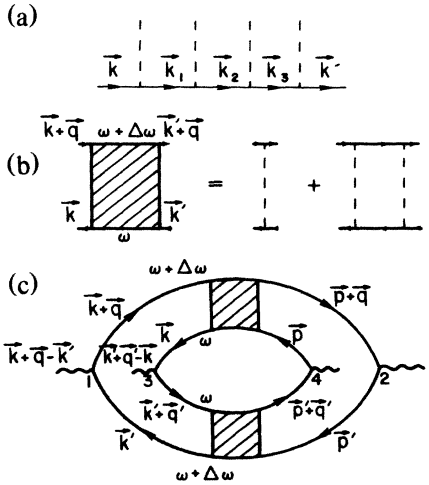

FIG. 1. (a) A diagram in the $G_\omega(\mathbf{k},\mathbf{k}')$ expansion before averaging. (b) The set of ladder diagrams arising in the average intensity calculation. (c) The diagram representing the intensity correlation function.

Let me stress that, in the weak-disorder limit, the ladder approximation is sufficient for calculating the total intensity [Eq. (8)]. It would not be, generally, sufficient if one requires the object $\langle G(\mathbf{k}_1,\mathbf{k}_2) \times G^*(\mathbf{k}_3,\mathbf{k}_4)\rangle$ for any value of momenta. This object contains more information than just the total intensity, to which it is related by Eq. (6). In particular, for certain values of momenta another set of diagrams (maximally crossed diagram) gives a contribution comparable to the ladder diagrams. This is the enhanced backscattering phenomenon, discussed in simple terms, e.g., by Akkermans and Maynard[16] and calculated for various geometries by Stephan[17] (for an early diagrammatic calculation see Barabanenkov[18]).

Next we calculate the intensity correlation function defined in Eq. (3). For this one has to draw two rings, representing the two intensities to be averaged, and to insert "ladder boxes," W, in all possible ways [e.g., Fig. 1(c)]. [The diagram with no connections between the two rings should not be counted, since it cancels the second term on the right-hand side of Eq. (3)]. The diagram in Fig. 1(c) is then multiplied by corresponding vertex factors, namely, $\exp[i(\mathbf{k}+\mathbf{q}-\mathbf{k}')\cdot\mathbf{r}]$ for vertex 1 and $\exp[i(\mathbf{k}'+\mathbf{q}'-\mathbf{k})\cdot\mathbf{r}']$ for vertex 3 (since $\mathbf{r}_0=0$, no factors for vertices 2 and 4 appear), and integrated over the momenta. It turns out that, in leading order with respect to $1/k_0 l$, the diagram in Fig. 1(c) is the only relevant diagram.

I discuss separately the two cases mentioned above.

(i) The source frequency is fixed, i.e., $\Delta\omega=0$. For this case the diagram gives

$$C(\mathbf{r},\mathbf{r}')=\langle I_\omega(\mathbf{r})\rangle\langle I_\omega(\mathbf{r}')\rangle(\sin k_0\Delta r/k_0\Delta r)^2\exp(-\Delta r/l), \tag{9}$$

where $\Delta r\equiv|\Delta\mathbf{r}|=|\mathbf{r}-\mathbf{r}'|$. For $\Delta r=0$, Eq. (9) reduces to the variance, $\langle\Delta I^2\rangle$, at point $\mathbf{r}$ and gives $\langle\Delta I^2\rangle=\langle I\rangle^2$. These are the large intensity fluctuations obtained previously by many workers.[1-3] A common inituitive argument for this result is based on the observation that the wave amplitude at point $\mathbf{r}$ is a sum of many contributions which correspond to different scattering processes. Assuming then random and uncorrelated phases for different processes, one can immediately calculate not only the moments of intensity but the entire probability distribution $P(I)=(1/\langle I\rangle)\times\exp(-I/\langle I\rangle)$ (it is useful to employ the analogy with a random walk in a plane[3]). For a smooth random potential (the parabolic approximation) this distribution, as well as corrections to it, was calculated directly from the wave equation by Dashen,[4] using a path-integral technique. The present diagrammatic method allows the derivation of this result also for the opposite case of a white-noise potential. Indeed, the nth moment of intensity, $\langle I^n\rangle$, is obtained by drawing n rings ($2n$ propagators) and inserting ladders between various pairs of propagators ($n!$ possibilities). Thus, $\langle I^n\rangle=n!\langle I\rangle^n$, which corresponds to the above written distribution for I. For $\Delta r\neq 0$, Eq. (9) tells us that the typical size of a speckle does not depend on the distance from the source and it is of the order of the (elastic) mean free path, or the phase coherence length, l. There is also some further intensity modulation within a speckle (the sine factor). As a result of diffusion-type propagation, a speckle is on the average isotropic. Qualitatively, these results should also hold for the slab geometry (commonly employed in experiments on light transmission and reflection), provided that the impinging light beam is sufficiently well collimated, namely, its width L is not large compared to l. In this case the "bulk" speckles, of size l, within the slabs should produce "plane" speckles of angular size λ/l (and larger) on a screen outside the slab (λ is the wavelength). On the other hand, for a wide impinging beam ($L>>l$), which so far seems to be the common experimental situation, L (and possibly, the slab thickness) is expected to be the relevant length.

(ii) The source frequency is changed by $\Delta\omega$ and the intensity change at point $\mathbf{r}$ is observed, i.e., $\Delta\mathbf{r}=0$. For this case the diagram in Fig. 1(c) gives

$$C(\mathbf{r};\Delta\omega)=\langle I(\mathbf{r})\rangle^2\exp\left[-\left(\frac{6r^2\Delta\omega}{lc_0}\right)^{1/2}\right]. \tag{10}$$

Thus, the intensity at point $\mathbf{r}$, as a function of frequency, will typically display a sequence of peaks and valleys separated by frequencies $\Delta\omega\simeq c_0 l/r^2$. This result is intuitively clear. Because of diffusion, the actual path length to the point $\mathbf{r}$ is $L\simeq r^2/l$. To produce a significant change in phase for this path, a change of order c_0/L, i.e., $c_0 l/r^2$, in the source frequency (or the electron energy) is needed.[7,9,10]

It is worthwhile mentioning that the correlation function $C(\mathbf{r},\mathbf{r}')$ is a somewhat simpler object than the correlation function for the electronic conductance studied in Refs. 10 and 11. It is true that the diagram in Fig. 1(c)—being related to the square of the density-density correlation function—does contain information about the diffusion coefficient (and thus conductance) fluctuations. However, in order to extract this information one has to consider, in addition to the limit $\Delta\omega\to 0$, the limit of zero external momenta (i.e., the momenta assigned to the wavy lines). The diagram then develops a strong singularity (fourth power of momentum in the denominator), which produces the universal conductance fluctuations.[10,11,19]

I am indebted to P. W. Anderson for introducing me to the subject of large intensity fluctuations and for many discussions and explanations in the course of this work. I am also grateful to P. A. Lee for an important discussion at an early stage of this work and to I. Affleck for numerous useful conversations involving some technical details. Useful discussions with S. John, A. Khurana, and S. Trugman are gratefully acknowledged. Thanks are also due to R. Fish for drawing my attention to Ref. 4. The research was supported by the National Science Foundation under Grant No. DMR 8518163.

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[1]A. Ishimaru, *Wave Propagation and Scattering in Random Media* (Academic, New York, 1978).

[2]V. I. Tatarskii and V. U. Zavortnyi, in *Strong Fluctuations in Light Propagation in a Randomly Inhomogeneous Medium,* edited by E. Wolf, Progress in Optics, Vol. 18 (North-Holland, Amsterdam, 1980), p. 204.

[3]J. W. Goodman, J. Opt. Soc. Am. **66**, 1145 (1976).

[4]R. Dashen, J. Math. Phys. **20**, 894 (1979).

[5]Y. Kuga and A. Ishimaru, J. Opt. Soc. Am. A **1**, 831 (1984); M. P. van Albada and A. Langendijk, Phys. Rev. Lett. **55**, 2692 (1985); P. E. Wolf and G. Maret, Phys. Rev. Lett. **55**, 2696 (1985).

[6]S. Etemad, R. Thompson, and M. J. Andrejco, Phys. Rev. Lett. **57**, 575 (1986).

[7]A. Genack, unpublished.

[8]For an extensive review on mesoscopic systems, see Y. Imry, in *Directions in Condensed Matter Physics,* edited by G. Grinstein and E. Mazenko (World Scientific, Singapore, 1986).

[9]A. D. Stone and Y. Imry, Phys. Rev. Lett. **56**, 189 (1986).

[10]P. A. Lee and A. D. Stone, Phys. Rev. Lett. **55**, 1622 (1985).

[11]B. L. Al'tshuler and D. E. Khmelnitskii, Pis'ma Zh. Eksp. Teor. Fiz. **42**, 201 (1985) [JETP Lett. **42**, 359 (1985)].

[12]S. Feng, P. A. Lee, and A. D. Stone, Phys. Rev. Lett. **56**, 1960 (1986); B. L. Al'tshuler and B. Z. Spivak, Pis'ma Zh. Eksp. Teor. Fiz. **42**, 363 (1985) [JETP Lett. **42**, 4477 (1985)].

[13]A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Physics* (Prentice-Hall, Englewood Cliffs, 1963).

[14]U. Frish, in *Probabilistic Methods in Applied Mathematics,* edited by A. T. Bharucha-Reid (Academic, New York, 1968), Vol. 1, p. 75.

[15]See, e.g., H. Fukuyama, Prog. Theor. Phys., Suppl. **84**, 47 (1985).

[16]E. Akkermans and R. Maynard, J. Phys. (Paris), Lett. **46**, L1045 (1985).

[17]M. J. Stephan, Phys. Rev. Lett. **56**, 1809 (1986); M. J. Stephan and G. Cwilich, unpublished.

[18]Yu. N. Barabanenkov, Izv. Vyssho. Uchebn. Zaved. Radiofiz. **16**, 88 (1973).

[19]It is interesting to note that the object $\Gamma(R) \equiv \int d\Omega\, R^2 I(R,\Omega)$ (i.e., the intensity integrated over a spherical shell) exhibits fluctuations similar to the "universal conductance fluctuations," namely, $\langle \Gamma(R) \rangle \sim R$ and $\langle \Delta\Gamma^2(R) \rangle \sim$ const, for large R.